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Off-Shell $d = 5$ Supergravity Coupled to a Matter-Yang-Mills System

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Abstract

We present an off-shell formulation of a matter-Yang-Mills system coupled to supergravity in five-dimensional space-time. We give an invariant action for a general system of vector multiplets and hypermultiplets coupled to supergravity as well as the supersymmetry transformation rules. All the auxiliary fields are retained, so that the supersymmetry transformation rules remain unchanged even when the action is changed.

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§1. Introduction

It is a revolutionary and interesting idea that our four-dimensional world may be a '3-brane' embedded in a higher-dimensional spacetime. In order to investigate various problems seriously in such brane world scenarios, however, we need to understand supergravity theory in five dimensions. [1), 2)]

We are interested in five-dimensional space-time since it provides us with the simplest case in which we have a single extra dimension. Also, in a more realistic situation, it is believed that M-theory, whose low energy effective theory is described by eleven-dimensional supergravity, is compactified on Calabi-Yau 3-folds and that it can then be described by effective five-dimensional supergravity theories. [3), 4)]

In the framework of the 'on-shell formulation' (that is, the formulation in which there are no auxiliary fields and hence the supersymmetry algebra closes only on-shell), Günayden, Sierra and Townsend [5)] (GST) proposed such a five-dimensional supergravity theory which contains general Yang-Mills/Maxwell vector multiplets. Their work was extended recently by Ceresole and Dall'Agata [6)] to a rather general system containing also tensor (linear) multiplets and hypermultiplets.

However, in various problems we need an off-shell formulation containing the auxiliary fields with which the supersymmetry transformation laws are made system-independent and the algebra closes without using equations of motion. For instance, Mirabelli and Peskin [7)] were able to give a simple algorithm based on an off-shell formulation for finding how to couple bulk fields to boundary fields in a work in which they considered a five-dimensional super Yang-Mills theory compactified on S^1/Z_2. They clarified how supersymmetry breaking occurring on one boundary is communicated to another. Moreover, if we wish to study problems by adding D-branes to such a system, then, without an off-shell formulation, we must find a new supersymmetry transformation rule of the bulk fields each time that we add new branes, since the supersymmetry transformation law in the on-shell formulation depends on the Lagrangian of the system.

A 5D supergravity tensor calculus for constructing an off-shell formulation has been given by Zucker. [8)] In a previous paper, [9)] which we refer to as I henceforth, we derived a more complete tensor calculus using dimensional reduction from 6D superconformal tensor calculus. [10)] Tensor calculus gives a set of rules in off-shell supergravity: i) transformation laws of the various types of supermultiplets; ii) composition laws of multiplets from multiplets; and iii) invariant action formulas. In this paper, we construct an action for a general system of vector multiplets and hypermultiplets coupled to supergravity based on the tensor calculus presented in I. This is, in principle, a straightforward task (containing no trial-and-error

steps). Nevertheless it requires considerable computations to simplify the form of the action and transformation laws; in particular, we must perform a change of variables in order to make the Rarita-Schwinger term canonical by solving the mixing between the gravitino and matter fermion fields.

In §2, we present an invariant action for the system of vector multiplets. Although a certain index must be restricted to be of an Abelian group in order for the tensor calculus formulas to be applicable, we find that the action can in fact be generalized to non-Abelian cases by a slight modification. The action for the system of hypermultiplets is next given in §3, where the mass term is also included. In §4, we combine these two systems and make a first step to simplifying the form of the total action. In §5, we fix the dilatation gauge and perform a change of variables to obtain the final form of the action, in which both the Einstein and Rarita-Schwinger terms take canonical forms. This gauge fixing and change of variables modify the supersymmetry transformation into a combination of the original supersymmetry and other transformations, which are carried out in §6. In §7, we give comments on i) the relation to the independent variables used in GST, ii) compensator components in the hypermultiplets, iii) the gauging of $SU(2)_R$ and $U(1)_R$, and iv) the scalar potential term in the action. We conclude in §8. Appendix A gives a technical proof for the existence of a representation matrix. In Appendix B, we explicitly show how the manifold $U(2,n)/U(2)\times U(n)$ is obtained as a target space of the hypermultiplet scalar fields for the case of two (quaternion) compensators.[14)]

In this paper, we do not give the tensor calculus formulas presented in our previous paper I, but we freely refer to the equations given there. For instance, (I 2·3) denotes Eq. (2·3) in I. For clarity, however, we list in Table I the field contents of the Weyl multiplets, vector multiplets and hypermultiplets, which we deal with in this paper. (The dilatation gauge field b_μ and spin connection ω_μ^{ab} are also listed, although they are dependent fields.) The notation is the same as in I, with one exception: Here we use χ^i to denote the auxiliary fermion component of the Weyl multiplet denoted by $\tilde{\chi}^i$ in I.

§2. Vector multiplet action

Let $\boldsymbol{V}^I \equiv (M^I,\, W_\mu^I,\, \Omega^{I\,i},\, Y^{I\,ij})$ $(I=1,2,\cdots,n)$ be vector multiplets of a gauge group G, which we assume to be given generally by a direct product of simple groups G_i and $U(1)$ groups:

$$G=\prod_i G_i \times \prod_x U(1)_x. \qquad (G_i:\ \text{simple}) \tag{2·1}$$

Table I. Field contents of the multiplets.

field	type	restrictions	$SU(2)$	Weyl-weight
		Weyl multiplet		
${e_\mu}^a$	boson	fünfbein	$\mathbf{1}$	-1
ψ^i_μ	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$-\frac{1}{2}$
V^{ij}_μ	boson	$SU(2)$ gauge field $V^{ij}_\mu = V^{ji}_\mu = V^*_{\mu ij}$	$\mathbf{3}$	0
A_μ	boson	gravi-photon $A_\mu = e^z_5 e^5_\mu$	$\mathbf{1}$	0
α	boson	'dilaton' $\alpha = e^z_5$	$\mathbf{1}$	1
t^{ij}	boson	$t^{ij} = t^{ji} = t^*_{ij}$ $(= -V^{ij}_5)$	$\mathbf{3}$	1
v_{ab}	boson	real tensor $v_{ab} = -v_{ba}$	$\mathbf{1}$	1
χ^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
C	boson	real scalar	$\mathbf{1}$	2
b_μ	boson	$\boldsymbol{D}$ gauge field $b_\mu = \alpha^{-1}\partial_\mu\alpha$	$\mathbf{1}$	0
${\omega_\mu}^{ab}$	boson	spin connection	$\mathbf{1}$	0
		Vector multiplet		
W_μ	boson	real gauge field	$\mathbf{1}$	0
M	boson	real scalar, $M = -W_5$	$\mathbf{1}$	1
Ω^i	fermion	$SU(2)$-Majorana	$\mathbf{2}$	$\frac{3}{2}$
Y_{ij}	boson	$Y^{ij} = Y^{ji} = Y^*_{ij}$	$\mathbf{3}$	2
		Hypermultiplet		
$\mathcal{A}^\alpha_i$	boson	$\mathcal{A}^i_\alpha = \varepsilon^{ij}\mathcal{A}^\beta_j\rho_{\beta\alpha} = -(\mathcal{A}^\alpha_i)^*$	$\mathbf{2}$	$\frac{3}{2}$
ζ^α	fermion	$\bar\zeta^\alpha \equiv (\zeta_\alpha)^\dagger\gamma_0 = \zeta^{\alpha\mathrm{T}}C$	$\mathbf{1}$	2
$\mathcal{F}^\alpha_i$	boson	$\mathcal{F}^i_\alpha = -(\mathcal{F}^\alpha_i)^*$	$\mathbf{2}$	$\frac{5}{2}$

The structure constant ${f_{IJ}}^K$ of G, $[t_I, t_J] = -{f_{IJ}}^K t_K$, is nonvanishing only when I, J and K all belong to a common simple factor group G_i, and then it is the same as the structure constant of the simple group G_i. The gauge coupling constants can, of course, be different for each factor group G_i and $U(1)_x$, but for simplicity, we write the G transformation of

M^I, for instance, in the form $\delta_G(\theta)M^I = g[\theta, M]^I = -g f_{JK}{}^I \theta^J M^K$. The quantity g here, therefore, should be understood as representing the coupling constant g_i of G_i when $I \in G_i$. (M here in $[\theta, M]$ denotes a matrix such that $M = M^I t_I$.)

In addition to these $\boldsymbol{V}^I$ $(I = 1, 2, \cdots, n)$, we have a special vector multiplet called the 'central charge vector multiplet', which consists of the dilaton $\alpha = e_5^z$ and the gravi-photon $A_\mu = e_5^z e_\mu^5$ among the Weyl multiplet:

$$\boldsymbol{V}^{I=0} = (M^{I=0},\, W_\mu^{I=0},\, \varOmega^{I=0\,i},\, Y^{I=0\,ij}) \equiv (\alpha,\, A_\mu,\, 0,\, 0). \tag{2·2}$$

We henceforth extend the group index I to run from 0 to n and use $I = 0$ to denote this central charge vector multiplet as written here. Corresponding to this extension, the gauge group G should also be understood to include the central charge $\boldsymbol{Z}$ as one of the Abelian $U(1)_x$ factor groups. Note that the fermion and auxiliary field components of this multiplet are zero: $\varOmega^{I=0} = Y^{I=0} = 0$. Thus the number of scalar and vector components is each $n+1$, while the number of $\varOmega$ and Y components is each n, at this stage. (Below the number of scalar components is reduced by 1 through $\boldsymbol{D}$-gauge fixing.)

In I, we show that we can construct a linear multiplet $\boldsymbol{L} = (L^{ij},\, \varphi^i,\, E_a,\, N)$, denoted by $f(\boldsymbol{V})$, from vector multiplets $\boldsymbol{V}^I$ using any *homogeneous quadratic* polynomial in M^I,

$$f(M) = \tfrac{1}{2} f_{IJ} M^I M^J, \tag{2·3}$$

where I, J run from 0. The vector component E_a of a linear multiplet is subject to a 'divergenceless' constraint, and it can be replaced by an unconstrained anti-symmetric tensor (density) field $E^{\mu\nu}$ when $\boldsymbol{L}$ is completely neutral under G. The explicit expression for the components of this multiplet, $\boldsymbol{L} = f(\boldsymbol{V})$, L^{ij}, φ^i, E_a, N and $E^{\mu\nu}$, in terms of those of $\boldsymbol{V}^I$ is given in Eqs. (I 5·3) and (I 5·5). We also have the V-L action formula in Eq. (I 5·7), which gives an invariant action for any pair consisting of an Abelian vector multiplet $\boldsymbol{V} = (M, W_\mu, \varOmega^i, Y^{ij})$ and a linear multiplet $\boldsymbol{L} = (L^{ij},\, \varphi^i,\, E_a\,(\text{or } E^{\mu\nu}),\, N)$:

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm VL} = {}& Y^{ij}L_{ij} + 2i\bar{\varOmega}\varphi + 2i\bar{\psi}_a^i\gamma^a\varOmega^j L_{ij} + \tfrac{1}{2}M(N - 2i\bar{\psi}_b\gamma^b\varphi - 2i\bar{\psi}_a^i\gamma^{ab}\psi_b^j L_{ij}) \\ & - \tfrac{1}{2}W_a(E^a - 2i\bar{\psi}_b\gamma^{ba}\varphi + 2i\bar{\psi}_b^i\gamma^{abc}\psi_c^j L_{ij})\,. \end{aligned} \tag{2·4}$$

This formula is valid only when the liner multiplet $\boldsymbol{L}$ carries no gauge group charges or is charged only under the abelian group of this vector multiplet $\boldsymbol{V}$. When the linear multiplet carries no charges, the constrained component E^a can be replaced by the unconstrained anti-symmetric tensor $E^{\mu\nu}$, and the action formula (2·4) can be rewritten in a simpler form:

$$\begin{aligned} e^{-1}\mathcal{L}_{\rm VL} = {}& Y^{ij}L_{ij} + 2i\bar{\varOmega}\varphi + 2i\bar{\psi}_a^i\gamma^a\varOmega^j L_{ij} + \tfrac{1}{2}M(N - 2i\bar{\psi}_b\gamma^b\varphi - 2i\bar{\psi}_a^i\gamma^{ab}\psi_b^j L_{ij}) \\ & + \tfrac{1}{4}e^{-1}F_{\mu\nu}(W)E^{\mu\nu}. \end{aligned} \tag{2·5}$$

Now we use this invariant action formula (2·5) to construct a general action for our set of vector multiplets $\{\boldsymbol{V}^I\}$. Since this formula applies only to an Abelian vector multiplet $\boldsymbol{V}$, we first choose all the *Abelian* vector multiplets $\{\boldsymbol{V}^A\}$ from $\{\boldsymbol{V}^I\}$, and, for each abelian index A we prepare a G-invariant quadratic polynomial $f_A(M)$ to construct a neutral linear multiplet $\boldsymbol{L}_A = f_A(\boldsymbol{V})$ using Eqs. (I 5·3) and (I 5·5). We apply the V-L action formula (2·5) to these pairs of $\boldsymbol{V}^A$ and $\boldsymbol{L}_A = f_A(\boldsymbol{V})$ and sum over all the Abelian indices A. Then we rewrite super-covariantized quantities like $\hat{G}_{ab}(W)$, $\hat{\mathcal{D}}_a M^I$, $\hat{\mathcal{D}}_a \Omega^I$, etc., as non-supercovariantized quantities:

$$
\begin{aligned}
&\hat{G}^I_{ab}(W) = G^I_{ab}(W) + 4i\bar{\psi}_{[a}\gamma_{b]}\Omega^I, \\
&\hat{\mathcal{D}}_a \Omega^{Ii} = \mathcal{D}_a\Omega^{Ii} + (\tfrac{1}{4}\gamma\cdot G^I(W) + \tfrac{1}{2}\not{\mathcal{D}}M^I - Y^I)\psi^i_a + i\gamma^{bc}\psi^i_a(\bar{\psi}_b\gamma_c\Omega^I) - i\gamma^b\psi^i_a(\bar{\psi}_b\Omega^I), \\
&\hat{\mathcal{D}}_a M^I = \mathcal{D}_a M^I - 2i\bar{\psi}_a\Omega^I. \qquad (2·6)
\end{aligned}
$$

Here, $\mathcal{D}_\mu$ is the usual covariant derivative, which is covariant only with respect to the homogeneous transformations $\boldsymbol{M}_{ab}$, $\boldsymbol{U}_{ij}$, $\boldsymbol{D}$ and $\boldsymbol{G}$. Then, interestingly, many cancellations occur, and the resultant expression is no more complicated than that written with supercovariantized quantities. Using the notation

$$
f_A \equiv f_A(M) = \tfrac{1}{2}f_{A,JK}M^J M^K, \quad f_{A,J} \equiv \frac{\partial f_A}{\partial M^J}\,, \quad f_{A,JK} \equiv \frac{\partial^2 f_A}{\partial M^J \partial M^K}\,, \qquad (2·7)
$$

the result is given by

$$
\begin{aligned}
&Y^{Aij}L_{Aij} + 2i\bar{\Omega}^A\varphi_A + 2i\bar{\psi}^i_a\gamma^a\Omega^{Aj}L_{Aij} \\
&\quad = f_A\left(2Y^A\cdot t - 4i\bar{\psi}\cdot\gamma t\Omega^A - 8i\bar{\Omega}^A\chi\right) \\
&\qquad + f_{A,J}\begin{pmatrix} -Y^A\cdot Y^J + 2i\bar{\Omega}^A(\not{\mathcal{D}} - \tfrac{1}{2}\gamma\cdot v + t)\Omega^J + i\bar{\Omega}^A\gamma^a(\tfrac{1}{2}\gamma\cdot G + \not{\mathcal{D}}M)^J\psi_a \\ -2(\bar{\Omega}^A\gamma^a\gamma^{bc}\psi_a)(\bar{\psi}_b\gamma_c\Omega^J) + 2(\bar{\Omega}^A\gamma^a\gamma^b\psi_a)(\bar{\psi}_b\Omega^J) \end{pmatrix} \\
&\qquad + f_{A,JK}\begin{pmatrix} -2i\bar{\Omega}^A(\tfrac{1}{4}\gamma\cdot G - \tfrac{1}{2}\not{\mathcal{D}}M + Y)^J\Omega^K - i\bar{\Omega}^J Y^A\Omega^K \\ +2(\bar{\Omega}^A\gamma^{ab}\Omega^J)(\bar{\psi}_a\gamma_b\Omega^K) + 2(\bar{\Omega}^A\gamma^a\Omega^J)(\bar{\psi}_a\Omega^K) \\ +2(\bar{\psi}^i\cdot\gamma\Omega^{Aj})(\bar{\Omega}^J_{(i}\Omega^K_{j)}) \end{pmatrix}, \qquad (2·8)
\end{aligned}
$$

$$
\begin{aligned}
&\tfrac{1}{2}M^A(N_A - 2i\bar{\psi}_b\gamma^b\varphi_A - 2i\bar{\psi}^i_a\gamma^{ab}\psi^j_b L_{Aij}) + \tfrac{1}{4}e^{-1}F^A_{\mu\nu}(W)E^{\mu\nu}_A \\
&\quad = \tfrac{1}{2}f_{A,J}\mathcal{D}^a M^A(\mathcal{D}_a M^J - 2i\bar{\psi}_b\gamma^a\gamma^b\Omega^J) \\
&\qquad + \tfrac{1}{2}f_A M^A\begin{pmatrix} -4C - 16t\cdot t - \tfrac{1}{2\alpha}F_{ab}(A)(4v^{ab} + i\bar{\psi}_c\gamma^{abcd}\psi_d) \\ +8i\bar{\psi}\cdot\gamma\chi + 4i\bar{\psi}_a\gamma^{ab}t\psi_b \end{pmatrix}
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{2}f_{A,J}M^A\begin{pmatrix} 4t\cdot Y^J-16i\bar{\Omega}^J\chi-8i\bar{\psi}\cdot\gamma t\Omega^J+2ig[\bar{\Omega},\,\Omega]^J \\ -\tfrac{1}{2}G^{Jab}(W)(4v^{ab}+i\bar{\psi}_c\gamma^{abcd}\psi_d)\end{pmatrix}\\
&+\tfrac{1}{2}f_{A,JK}M^A\begin{pmatrix} -\tfrac{1}{4}G^J(W)\cdot G^K(W)+\tfrac{1}{2}\mathcal{D}_aM^J\mathcal{D}^aM^K-Y^J\cdot Y^K \\ +2i\bar{\Omega}^J(\not{\mathcal{D}}-\tfrac{1}{2}\gamma\cdot v+t)\Omega^K+i\bar{\psi}_a(\gamma\cdot G-2\not{\mathcal{D}}M)^J\gamma^a\Omega^K \\ +(\bar{\Omega}^J\gamma^{ab}\Omega^K)(\bar{\psi}_a\psi_b)-2(\bar{\psi}^i_a\gamma^{ab}\psi^j_b)(\bar{\Omega}^J_i\Omega^K_j) \\ -4(\bar{\psi}_a\gamma^{abc}\Omega^K)(\bar{\psi}_b\gamma_c\Omega^K)+4(\bar{\psi}_{[a}\gamma_{b]}\Omega^J)(\bar{\psi}^a\gamma^b\Omega^K) \\ -2(\bar{\psi}_a\Omega^J)(\bar{\psi}^a\Omega^K)\end{pmatrix}\\
&-\tfrac{1}{4}G^A_{\mu\nu}(W)\Big(f_A(4v^{\mu\nu}+i\bar{\psi}_\rho\gamma^{\mu\nu\rho\sigma}\psi_\sigma)+if_{A,JK}\bar{\Omega}^J\gamma^{\mu\nu}\Omega^K\\
&\qquad\qquad+f_{A,J}(G^{J\mu\nu}(W)-2i\bar{\psi}_\lambda\gamma^{\mu\nu}\gamma^\lambda\Omega^J)\Big)\\
&-e^{-1}\tfrac{1}{4}f_{A,JK}\epsilon^{\lambda\mu\nu\rho\sigma}W^A_\lambda F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W)\ . \qquad (2\cdot 9)
\end{aligned}
$$

Here and throughout this paper, we use the following convention for the $SU(2)$ triplet quantities X^{ij}, like t^{ij}, Y^{Iij} and V^{ij}_μ: If their $SU(2)$ indices are suppressed, they represent the matrix $X^i{}_j$, so that $X\psi^i$, when acting on an $SU(2)$ spinor ψ^i like Ω^{Ii}, represents $X^i{}_j\psi^j$, and, similarly, $X\psi_i=X_{ij}\psi^j$, as obtained by lowering the index i on both sides. $X\cdot Y$, for two triplets X and Y, represents $\mathrm{tr}(XY)=X^i{}_jY^j{}_i=-X^{ij}Y_{ji}=-X^{ij}Y_{ij}$, and $X\cdot X$ is also written X^2. For instance, $\bar{\Omega}^AY^J\Omega^K$ in the above represents $\bar{\Omega}^{Ai}Y^J\Omega^K_i=\bar{\Omega}^{Ai}Y^J_{ij}\Omega^{Kj}$.

The action is given by the sum of Eqs. (2·8) and (2·9), where the indices J and K run over the whole group G, while the (external) index A of $f_A(M)$ is restricted to run only over the Abelian subset of G. However, interestingly, this action can be shown to be *totally symmetric* with respect to the three indices A, J and K of $f_{A,JK}$ if J and K are also restricted to the Abelian indices. In view also of the fact that this action formula itself gives an invariant action, including the case of non-Abelian indices for J and K, we suspect that this action gives an invariant action even if we extend the the index A of $f_A(M)$ to I running over the whole group G. In that case, the function $f_I(M)$ for the indices I belonging to the non-Abelian factor groups G_i of G should, of course, be a function giving the adjoint representation of G_i to satisfy the G invariance, and the Chern-Simons term should also be generalized to the corresponding one. (A similar situation also exists in the 6D case. [10]) Then the product $M^If_I(M)$ becomes a general G-invariant *homogeneous cubic* polynomial in M, which, with a sign change, is called a 'norm function' and denoted $\mathcal{N}(M)$, following Günayden, Sierra and Townsend: [5])

$$
\mathcal{N}(M)\equiv c_{IJK}M^IM^JM^K\ (=-M^If_I(M)). \qquad (2\cdot 10)
$$

Here the coefficient c_{IJK} is totally symmetric with respect to the indices. Now the resul-

tant action is characterized solely by this cubic polynomial $\mathcal{N}(M)$, and we find the vector multiplet action

$$
\begin{aligned}
&e^{-1}\mathcal{L}_{\rm VL} \\
&\quad = +\tfrac{1}{2}\mathcal{N}\left(4C+16t\cdot t+\tfrac{1}{2\alpha}F_{ab}(A)(4v^{ab}+i\bar\psi_c\gamma^{abcd}\psi_d)-8i\bar\psi\cdot\gamma\chi-4i\bar\psi_a\gamma^{ab}t\psi_b\right) \\
&\quad\quad -\mathcal{N}_I\left(\begin{array}{c} 2t\cdot Y^I-8i\bar\Omega^I\chi-4i\bar\psi\cdot\gamma t\Omega^I+ig[\bar\Omega,\,\Omega]^I \\ -G^I_{ab}(W)(v^{ab}+\tfrac{i}{4}\bar\psi_c\gamma^{abcd}\psi_d) \end{array}\right) \\
&\quad\quad -\tfrac{1}{2}\mathcal{N}_{IJ}\left(\begin{array}{c} -\tfrac{1}{4}G^I(W)\cdot G^J(W)+\tfrac{1}{2}\mathcal{D}_aM^I\mathcal{D}^aM^J-Y^I\cdot Y^J \\ +2i\bar\Omega^I(\not{\mathcal{D}}-\tfrac{1}{2}\gamma\cdot v+t)\Omega^J+i\bar\psi_a(\gamma\cdot G(W)-2\not{\mathcal{D}}M)^I\gamma^a\Omega^J \\ -2(\bar\Omega^I\gamma^a\gamma^{bc}\psi_a)(\bar\psi_b\gamma_c\Omega^J)+2(\bar\Omega^I\gamma^a\gamma^b\psi_a)(\bar\psi_b\Omega^J) \end{array}\right) \\
&\quad\quad -\mathcal{N}_{IJK}\left(\begin{array}{c} -i\bar\Omega^I(\tfrac{1}{4}\gamma\cdot G(W)+Y)^J\Omega^K \\ +\tfrac{2}{3}(\bar\Omega^I\gamma^{ab}\Omega^J)(\bar\psi_a\gamma_b\Omega^K)+\tfrac{2}{3}(\bar\psi^i\cdot\gamma\Omega^{Ij})(\bar\Omega^J_{(i}\Omega^K_{j)}) \end{array}\right) \\
&\quad\quad + e^{-1}\mathcal{L}_{\text{C-S}}\ ,
\end{aligned}
\tag{2·11}
$$

where $\mathcal{N}_I=\partial\mathcal{N}/\partial M^I$, $\mathcal{N}_{IJ}=\partial^2\mathcal{N}/\partial M^I\partial M^J$, etc., and $\mathcal{L}_{\text{C-S}}$ is the Chern-Simons term:

$$
\begin{aligned}
\mathcal{L}_{\text{C-S}} = \tfrac{1}{8}c_{IJK}\epsilon^{\lambda\mu\nu\rho\sigma}W^I_\lambda\Big(&F^J_{\mu\nu}(W)F^K_{\rho\sigma}(W)+\tfrac{1}{2}g[W_\mu,W_\nu]^JF^K_{\rho\sigma}(W) \\
&+\tfrac{1}{10}g^2[W_\mu,W_\nu]^J[W_\rho,W_\sigma]^K\Big)\,.
\end{aligned}
\tag{2·12}
$$

We have checked the supersymmetry invariance of this action for general non-Abelian cases as follows. When the gauge coupling g is set equal to zero, the action reduces to one with the same form as that for the Abelian case, and thus the invariance is guaranteed by the above derivation. When g is switched on, the covariant derivative $\mathcal{D}_\mu$ comes to include the G-covariantization term $-g\delta_G(W_\mu)$, and the field strength $F_{\mu\nu}(W)$ comes to include the non-Abelian term $-g[W_\mu,W_\nu]$. We, however, can use the variables $\mathcal{D}_\mu\phi$ ($\phi=M^I$, Ω^I) and $F_{\mu\nu}(W)$ as they stand in the action and in the supersymmetry transformation laws, keeping these g-dependent terms implicit inside of them. Then, we have only to keep track of *explicitly* g-dependent terms and make sure that these terms vanish in the supersymmetry transformation of the action. The explicitly g-dependent terms in the action are only the term $-ig\mathcal{N}_I[\bar\Omega,\Omega]^I$, aside from those in the Chern-Simons term. The Chern-Simons term is special because it contains the gauge field W^I_μ explicitly, and its supersymmetry transformation as a whole yields no explicit g-dependent terms, as we show below. In the supersymmetry transformations $\delta\phi$, explicitly g-dependent terms do not appear for $\phi=M^I$, Ω^I, $G^I_{\mu\nu}(W)$ or $F^I_{\mu\nu}(W)$, but appear only in δY^{Iij}, $\delta(\mathcal{D}_\mu M^I)$ and $\delta(\mathcal{D}_\mu\Omega^I)$. (For the latter

two, the supersymmetry transformation of W_μ contained implicitly in $\mathcal{D}_\mu$ produces additional explicitly g-dependent terms). It is easy to see that all these g-dependent terms cancel out in the transformation of the action.

In carrying out such computations, it is convenient to use a matrix notation to represent the norm function $\mathcal{N}$. One can show that, for any G-invariant $\mathcal{N}(M) = c_{IJK}M^I M^J M^K$, there is a set of hermitian matrices $\{ T_I \}$ which satisfies

$$c_{IJK} = \frac{1}{6}\,\mathrm{tr}(T_I\,\{T_J, T_K\}) \tag{2·13}$$

and gives a representation of G up to normalization constants c_i for each simple factor group G_i; that is, the rescaled matrices $t_I \equiv iT_I/c_{[I]}$, where $c_{[I]} = c_i$ for $I \in G_i$ and $c_{[I]} = 1$ for $I \in U(1)_x$, satisfy

$$[t_I,\, t_J] = -f_{IJ}{}^K t_K. \tag{2·14}$$

In Appendix A, we give a simple example of the representation of G which realizes these properties. Using the matrix notation $\tilde{X} \equiv X^I T_I$, we have

$$\begin{aligned}
&\mathcal{N} \equiv c_{IJK}M^I M^J M^K = \tfrac{1}{3}\,\mathrm{tr}(\tilde{M}^3), \\
&\mathcal{N}_I X^I = \mathrm{tr}(\tilde{X}\tilde{M}^2), \qquad \mathcal{N}_{IJ}X^I Y^J = \mathrm{tr}(\tilde{X}\{\tilde{Y}, \tilde{M}\}), \\
&\mathcal{N}_{IJK}X^I Y^J Z^K = \mathrm{tr}(\tilde{X}\{\tilde{Y}, \tilde{Z}\}).
\end{aligned} \tag{2·15}$$

With these expressions, we can simply use cyclic identities for the trace instead of referring to various cumbersome identities for c_{IJK} resulting from its G-invariance property. Note the difference from the ordinary matrix notation $X \equiv X^I t_I$: In the present case we have the relations $\widetilde{[X, Y]} = [X, Y]^I T_I = -f_{IJ}{}^K X^I Y^J T_K = [\tilde{X}, Y] = [X, \tilde{Y}]$, since $f_{IJ}{}^K$ is nonvanishing only when I, J, K belong to a common simple factor group G_i.

Using this matrix notation for the gauge field W^I_μ and the field strength $F^I_{\mu\nu}$, we can define the matrix-valued 1-form as $\tilde{W} \equiv \tilde{W}_\mu dx^\mu$ and the 2-form as $\tilde{F} \equiv \frac{1}{2}\tilde{F}_{\mu\nu}dx^\mu dx^\nu = d\tilde{W} - g\widetilde{W^2}$ (where $g\widetilde{W^2} = g\{\tilde{W}, W\}/2$), with which the Chern-Simons term (2·12) can be rewritten in the form

$$\int \mathcal{L}_{\text{C-S}}\, d^5x = \int \frac{1}{6}\,\mathrm{tr}\left(\tilde{W}\tilde{F}\tilde{F} + \frac{1}{4}\{\tilde{W},\, g\widetilde{W^2}\}\tilde{F} + \frac{1}{10}\tilde{W}\, g\widetilde{W^2}\, g\widetilde{W^2}\right). \tag{2·16}$$

For an arbitrary variation of W^I_μ, i.e., $\delta\tilde{W} = \tilde{X}$ in the matrix-valued 1-form notation, we find $\delta\tilde{F} = d\tilde{X} - g\widetilde{\{W, X\}}$. Using the Bianchi identity $D\tilde{F} = d\tilde{F} - g\widetilde{[W, F]} = 0$ and the properties $g\widetilde{\{W, X\}} = \{g\tilde{W}, X\} = \{gW, \tilde{X}\}$, $g\widetilde{[W, F]} = [g\tilde{W}, F] = [gW, \tilde{F}]$ and $[g\widetilde{W^2}, W] = [gW^2, \tilde{W}] = g\widetilde{[W^2, W]} = 0$, we can show

$$\delta\,\mathrm{tr}\left(\tilde{W}\tilde{F}\tilde{F}\right) = \mathrm{tr}\left(3\tilde{F}\tilde{F}\tilde{X} - \{\tilde{F},\, g\widetilde{W^2}\}\tilde{X}\right),$$

$$\delta\,\mathrm{tr}\left(\{\tilde W,\, g\widetilde{W^2}\}\tilde F\right)=\mathrm{tr}\left(4\{\tilde F,\, g\widetilde{W^2}\}\tilde X-2g\widetilde{W^2}\,g\widetilde{W^2}\tilde X\right),$$
$$\delta\,\mathrm{tr}\left(\tilde W\, g\widetilde{W^2}\, g\widetilde{W^2}\right)=\mathrm{tr}\left(5g\widetilde{W^2}\, g\widetilde{W^2}\tilde X\right), \tag{2·17}$$

so that the variation of the Chern-Simons term indeed gives no explicitly g-dependent term, as claimed above:

$$\delta\int \mathcal{L}_{\text{C-S}}\, d^5x=\int \frac{1}{2}\,\mathrm{tr}(\tilde F\tilde F\delta\tilde W)\,. \tag{2·18}$$

§3. Hypermultiplet action

Now let $\boldsymbol{H}^\alpha=(\mathcal{A}_i^\alpha,\,\zeta^\alpha,\,\mathcal{F}_i^\alpha)$ $(\alpha=1,2,\cdots,2r)$ be a set of hypermultiplets which belongs to a representation ρ of the gauge group G. Under the G transformation it transforms as $\delta_G(\theta)\boldsymbol{H}^\alpha=\sum_{I=1}^n g\theta^I\rho(t_I)^\alpha{}_\beta\boldsymbol{H}^\beta$. The ordinary matrix notation used for the vector multiplet in the preceding section was, for instance, $M=M^It_I$, and the matrix t_I denoted an adjoint representation $\mathrm{ad}(t_I)$ of G. The representation ρ here can, of course, be different from the adjoint representation ad. However, to avoid cumbersome expressions, we simplify the matrix notation and write, e.g., $M\mathcal{A}_i^\alpha=M^\alpha{}_\beta\mathcal{A}_i^\beta$ to represent $\rho(M)^\alpha{}_\beta\mathcal{A}_i^\beta=M^I\rho(t_I)^\alpha{}_\beta\mathcal{A}_i^\beta$. (Note $M\mathcal{A}_{\alpha i}=M_{\alpha\beta}\mathcal{A}_i^\beta$.)

The invariant action for the hypermultiplets is derived in I from the action in 6D and is given by Eq. (I 4·11).* Again we rewrite the supercovariant derivative $\hat{\mathcal{D}}_\mu$ in terms of the usual covariant derivative $\mathcal{D}_\mu$, which is covariant only with respect $\boldsymbol{M}_{ab}$, $\boldsymbol{U}_{ij}$, $\boldsymbol{D}$ and $\boldsymbol{G}$. (Note that covariantization with respect to the central charge $\boldsymbol{Z}$ transformation is also taken out.) Then we obtain the following action for the kinetic term of the hypermultiplets:

$$\begin{aligned}
e^{-1}\mathcal{L}_{\text{kin}}={}&\mathcal{D}^a\mathcal{A}_i^{\bar\alpha}\mathcal{D}_a\mathcal{A}_\alpha^i-2i\bar\zeta^{\bar\alpha}\not{\mathcal{D}}\zeta_\alpha+\frac{i}{2\alpha}\bar\zeta^{\bar\alpha}\gamma\cdot F(A)\zeta_\alpha-i\bar\zeta^{\bar\alpha}\gamma\cdot v\zeta_\alpha\\
&+2ig\bar\zeta^{\bar\alpha}M_\alpha{}^\beta\zeta_\beta+\mathcal{A}_i^{\bar\alpha}(t+gM)^2\mathcal{A}_\alpha^i-4i\bar\psi_a^i\zeta_\alpha\gamma^b\gamma^a\mathcal{D}^a\mathcal{A}_i^{\bar\alpha}\\
&+\begin{pmatrix}
2i\bar\zeta_\alpha\gamma^{ab}\mathcal{R}_{ab}{}^i(Q)-8i\bar\zeta_\alpha\chi^i\\
+\frac{i}{\alpha}\bar\psi_a^i\gamma^{abc}\zeta_\alpha\hat F_{bc}(A)-4i\bar\psi^{ia}\gamma^b\zeta_\alpha v_{ab}+4i\bar\psi_{aj}\gamma^a\zeta_\alpha t^{ij}\\
-8ig\bar\Omega_\alpha^i{}^\beta\zeta_\beta+4ig\bar\psi_a^i\gamma^aM_\alpha{}^\beta\zeta_\beta
\end{pmatrix}\mathcal{A}_i^{\bar\alpha}\\
&-2i\bar\psi_a^{(i}\gamma^{abc}\psi_c^{j)}\mathcal{A}_j^{\bar\alpha}\mathcal{D}_b\mathcal{A}_{\alpha i}\\
&+\begin{pmatrix}
C+\frac{1}{4}\mathcal{R}(M)+\frac{i}{2}\bar\psi_a\gamma^{abc}\mathcal{R}_{bc}(Q)\\
-2i\bar\psi_a\gamma^a\chi+\frac{1}{8\alpha^2}\hat F(A)^2-v^2+2t\cdot t\\
-\frac{i}{4\alpha}\bar\psi_a\gamma^{abcd}\psi_b\hat F_{cd}(A)+i\bar\psi_a\psi_bv^{ab}-i\bar\psi_a^i\gamma^{ab}\psi_b^jt_{ij}
\end{pmatrix}\mathcal{A}^2
\end{aligned}$$

---

* This action can also be derived if we make a linear multiplet $\boldsymbol{L}=d_{\alpha\beta}\boldsymbol{H}^\alpha\times\boldsymbol{Z}\boldsymbol{H}^\beta$ from the hypermultiplets $\boldsymbol{H}^\alpha$ and their central-charge transforms $\boldsymbol{Z}\boldsymbol{H}^\beta$ by using the formula (I 5·6), and then apply the linear multiplet action formula (I 5·9) to it.

$$
\begin{aligned}
&+ 2gY^{ij}_{\alpha\beta}\mathcal{A}^{\bar\alpha}_i\mathcal{A}^{\beta}_j + 4ig\bar\psi^{(i}_a\gamma^a\Omega^{j)}_{\alpha\beta}\mathcal{A}^{\bar\alpha}_i\mathcal{A}^{\beta}_j \\
&+ 2ig\bar\psi^{(i}_a\gamma^{ab}\psi^{j)}_b\mathcal{A}^{\bar\alpha}_i M_\alpha{}^\beta\mathcal{A}_{\beta j} + (1 - A^aA_a/\alpha^2)\mathcal{F}^{\bar\alpha}_i\mathcal{F}^i_\alpha \\
&+ \bar\psi_a\gamma_b\psi_c\bar\zeta^{\bar\alpha}\gamma^{abc}\zeta_\alpha - \tfrac{1}{2}\bar\psi^a\gamma^{bc}\psi_a\bar\zeta^{\bar\alpha}\gamma_{bc}\zeta_\alpha \ ,
\end{aligned} \tag{3·1}
$$

where the contraction between a pair with a barred index $\bar\alpha$ and α is defined as

$$
\mathcal{A}^{\bar\alpha}_i\mathcal{A}_{\alpha j} \equiv \mathcal{A}^{\beta}_i d_\beta{}^\alpha\mathcal{A}_{\alpha j}, \qquad \mathcal{A}^2 \equiv \mathcal{A}^{\bar\alpha}_i\mathcal{A}^i_\alpha, \qquad \bar\zeta^{\bar\alpha}\zeta_\alpha \equiv \bar\zeta^\beta d_\beta{}^\alpha\zeta_\alpha, \tag{3·2}
$$

by using the G-invariant metric $d_\alpha{}^\beta$ introduced in Eqs. (I B·22) and (I B·23). This metric $d_\alpha{}^\beta$ is, in its standard form, diagonal and takes the values ± 1. [11] Note in the above that $(t+gM)^2\mathcal{A}^i_\alpha = t^i{}_k t^k{}_j\mathcal{A}^j_\alpha + 2gM_{\alpha\beta}t^i{}_j\mathcal{A}^{\beta j} + gM_{\alpha\gamma}gM^\gamma{}_\beta\mathcal{A}^{\beta i}$ with our present convention. The hypermultiplets can have masses, and the invariant action for the mass term is given by Eq. (I 4·14), which reads

$$
e^{-1}\mathcal{L}_{\rm mass} = m\eta^{\alpha\beta}\begin{pmatrix}
-A^a\mathcal{D}_a\mathcal{A}_{\alpha i}\mathcal{A}^i_\beta - (1 - A^aA_a/\alpha^2)\alpha\mathcal{F}_{\alpha i}\mathcal{A}^i_\beta \\
-2i\bar\psi^i_a\zeta_\alpha A^a\mathcal{A}_{\beta i} + \alpha\mathcal{A}_{i\alpha}(t+gM)\mathcal{A}^i_\beta \\
+i(-\alpha\bar\zeta_\alpha\zeta_\beta + A_a\bar\zeta_\alpha\gamma^a\zeta_\beta) \\
+2i\mathcal{A}_{\alpha i}(-\alpha\bar\psi^i_a\gamma^a\zeta_\beta + \bar\psi^i_a\gamma^{ab}\zeta_\beta A_b) \\
+i\mathcal{A}_{\alpha i}\mathcal{A}_{\beta j}(-\alpha\bar\psi^i_a\gamma^{ab}\psi^j_b + \bar\psi^i_a\gamma^{abc}\psi^j_c A_b)
\end{pmatrix}. \tag{3·3}
$$

(Note that m is a dimensionless parameter, and the actual mass is proportional to $m\langle\alpha\rangle$.) Here $\eta^{\alpha\beta}$ is a symmetric G-invariant tensor. [11] Interestingly, this mass term turns out to be automatically included in the previous kinetic term action (3·1), and it need not be considered separately, provided that we complete the square for the terms containing the auxiliary fields $\mathcal{F}^\alpha_i$ in $\mathcal{L}_{\rm kin} + \mathcal{L}_{\rm mass}$. (Essentially the same observation is made in Ref. 11).) Doing so, the $\mathcal{F}^\alpha_i$ terms become

$$
(1 - A^aA_a/\alpha^2)\tilde{\mathcal{F}}^{\bar\alpha}_i\tilde{\mathcal{F}}^i_\alpha \qquad \text{with} \quad \tilde{\mathcal{F}}^\alpha_i \equiv \mathcal{F}^\alpha_i + \tfrac{1}{2}m\alpha(d^{-1})_\gamma{}^\alpha\eta^{\gamma\beta}\mathcal{A}_{\beta i}\,, \tag{3·4}
$$

and then, all the other terms in $\mathcal{L}_{\rm mass}$ can be absorbed into the kinetic Lagrangian $\mathcal{L}_{\rm kin}$ if we extend the gauge index I of the generators t_I acting on the hypermultiplets to run also from 0 and introduce

$$
(gt_{I=0})^{\alpha\beta} \equiv \tfrac{1}{2}m(d^{-1})_\gamma{}^\alpha\eta^{\gamma\beta}, \tag{3·5}
$$

so that gW_μ in $\mathcal{D}_\mu$ and M are now understood to be

$$
\begin{aligned}
gW_\mu &= \sum_{I=1}^n W^I_\mu(gt_I) + A_\mu(gt_0)\,, \\
gM &= \sum_{I=1}^n M^I(gt_I) + \alpha(gt_0)\,.
\end{aligned} \tag{3·6}
$$

§4. First step in rewriting the action

Now, the invariant action for our Yang-Mills-matter system coupled to supergravity is given by the sum $\mathcal{L} = \mathcal{L}_{\rm VL}[(2\cdot11)] + \mathcal{L}_{\rm kin}[(3\cdot1)]$, where in $\mathcal{L}_{\rm kin}$ the $\mathcal{F}^2$ term is replaced by (3·4), and Eq. (3·6) is understood.

We first note that the auxiliary fields C and χ appear in the action $\mathcal{L}$ in the form of Lagrange multipliers:

$$C(\mathcal{A}^2 + 2\mathcal{N}) - 8i\bar{\chi}(\zeta + \Omega)\,, \tag{4·1}$$

where ζ_i and Ω_i are defined as

$$\zeta_i \equiv \mathcal{A}_i^{\bar{\alpha}}\zeta_\alpha = \mathcal{A}_i^\beta d_\beta{}^\alpha \zeta_\alpha, \qquad \Omega_i \equiv \mathcal{N}_I \Omega_i^I\,. \tag{4·2}$$

That is, $\mathcal{A}^2 = -2\mathcal{N}$ and $\zeta_i = -\Omega_i$ are equations of motion. Although we do not use equations of motion, we can rewrite the terms multiplied by $\mathcal{A}^2$, $\mathcal{A}^2 X$, as $-2\mathcal{N}X$ with the shift $C \to C + X$, and, similarly, we can rewrite the terms $\bar{X}\zeta$ as $-\bar{X}\Omega$ with the shift $\chi \to \chi + iX/8$. Using this, we replace all the terms containing the factor $\mathcal{A}^2$ and all the terms containing the factor $\zeta_i = \mathcal{A}_i^{\bar{\alpha}}\zeta_\alpha$ in $\mathcal{L}_{\rm kin}$ by those multiplied by $\mathcal{N}$ and by Ω_i, respectively.

When doing this, we also rewrite the covariant derivative $\mathcal{D}_\mu$ in the following form, separating the terms containing gauge fields b_μ $(= \alpha^{-1}\partial_\mu\alpha)$ and V_μ^{ij}:

$$\mathcal{D}_\mu = \nabla_\mu - \delta_D(b_\mu) - \delta_U(V_\mu^{ij}) - \delta_M(-2e_\mu{}^{[a}b^{b]}). \tag{4·3}$$

The last term appears because the spin connection ω_μ^{ab} contains the b_μ field as

$$\begin{aligned} \omega_\mu{}^{ab} &= \omega_\mu^{0\ ab} + i(2\bar{\psi}_\mu\gamma^{[a}\psi^{b]} + \bar{\psi}^a\gamma_\mu\psi^b) - 2e_\mu{}^{[a}b^{b]}\,, \\ \omega_\mu^{0\ ab} &\equiv -2e^{\nu[a}\partial_{[\mu}e_{\nu]}{}^{b]} + e^{\rho[a}e^{b]\sigma}e_\mu{}^c\partial_\rho e_{\sigma c}\,. \end{aligned} \tag{4·4}$$

Then, the covariant derivative ∇_μ is now covariant only with respect to local-Lorentz and group transformations, and the spin connection is that with b_μ set equal to 0:

$$\nabla_\mu = \partial_\mu - \delta_M(\omega_\mu^{ab}|_{b_\mu=0}) - \delta_G(W_\mu). \tag{4·5}$$

We perform this separation of the b_μ and V_μ^{ij} gauge fields also for $\mathcal{R}(M)$ and $\mathcal{R}_{ab}^i(Q)$. This separation also yields several terms proportional to $\mathcal{A}^2$ and ζ_i, which also can be rewritten as terms proportional to $\mathcal{N}$ and Ω_i with shifts of C and χ.

Thus, we finally define C' and χ' in terms of C and χ as follows:

$$\begin{aligned} C' = C &+ \tfrac{1}{4}\mathcal{R}(M) + \tfrac{i}{2}\bar{\psi}_a\gamma^{abc}\mathcal{R}_{bc}(Q) - 2i\bar{\psi}_a\gamma^a\chi + \tfrac{1}{8\alpha^2}\hat{F}(A)^2 \\ &- v^2 - \tfrac{i}{4\alpha}\bar{\psi}_a\gamma^{abcd}\psi_b\hat{F}_{cd}(A) + i\bar{\psi}_a\psi_b v^{ab} - i\bar{\psi}_a^i\gamma^{ab}\psi_b^j t_{ij} \end{aligned}$$

$$
\begin{aligned}
&+\frac{9}{4}b^2+\frac{5}{2}t\cdot t+\frac{3}{2}e^{-1}\nabla_\mu(eb^\mu)+\frac{1}{2}V_a^{ij}V_{ij}^a+i\bar\psi_b^i\gamma^{bac}\psi_c^jV_{a\,ij}\,,\\
\chi_i'={}&\chi_i-\frac{1}{4}\gamma^{ab}\mathcal{R}_{abi}(Q)+\frac{1}{8\alpha}\gamma^{abc}\psi_{ai}\hat F_{bc}(A)\\
&+\frac{1}{2}\gamma_b\psi_{ai}v^{ab}+\frac{1}{2}t\gamma\cdot\psi_i+\gamma^a\gamma^b(\frac{1}{2}V_b-\frac{3}{4}b_b)\psi_{ai}\,. \qquad (4\cdot6)
\end{aligned}
$$

We also separate and collect the terms containing $F_{ab}(A)$ and the auxiliary fields v^{ab}, V_μ^{ij}, t^{ij}, Y^{Iij}, $\mathcal{F}_\alpha^i$. Then the action $\mathcal{L}$ is found to take the following form at this stage:

$$
\begin{aligned}
\mathcal{L}={}&\mathcal{L}'_{\rm hyper}+\mathcal{L}'_{\rm vector}+\mathcal{L}_{\text{C-S}}+\mathcal{L}'_{\rm aux}\,,\\
e^{-1}\mathcal{L}'_{\rm hyper}={}&\nabla^a\mathcal{A}_i^{\bar\alpha}\nabla_a\mathcal{A}_\alpha^i-2i\bar\zeta^{\bar\alpha}(\nabla\!\!\!\!/+gM)\zeta_\alpha\\
&+\mathcal{A}_i^{\bar\alpha}(gM)^2{}_\alpha{}^\beta\mathcal{A}_\beta^i-4i\bar\psi_a^i\gamma^b\gamma^a\zeta_\alpha\nabla_b\mathcal{A}_i^{\bar\alpha}-2i\bar\psi_a^{(i}\gamma^{abc}\psi_c^{j)}\mathcal{A}_j^{\bar\alpha}\nabla_b\mathcal{A}_{\alpha i}\\
&+\mathcal{A}_i^{\bar\alpha}\Big(8ig\bar\Omega_{\alpha\beta}^i\zeta^\beta-4ig\bar\psi_a^i\gamma^aM_{\alpha\beta}\zeta^\beta\\
&\qquad+4ig\bar\psi_a^{(i}\gamma^a\Omega_{\alpha\beta}^{j)}\mathcal{A}_j^\beta-2ig\bar\psi_a^{(i}\gamma^{ab}\psi_b^{j)}M_{\alpha\beta}\mathcal{A}_j^\beta\Big)\\
&+\bar\psi_a\gamma_b\psi_c\bar\zeta^{\bar\alpha}\gamma^{abc}\zeta_\alpha-\frac{1}{2}\bar\psi^a\gamma^{bc}\psi_a\bar\zeta^{\bar\alpha}\gamma_{bc}\zeta_\alpha\,,\\
e^{-1}\mathcal{L}'_{\rm vector}={}&\mathcal{N}\left(-\frac{1}{2}\mathcal{R}(M)|_{b=0}-2i\bar\psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho+(\bar\psi_a\psi_b)(\bar\psi_c\gamma^{abcd}\psi_d+\bar\psi^a\psi^b)\right)\\
&-\mathcal{N}_I\begin{pmatrix}-4i\bar\Omega^I\gamma^{\mu\nu}\nabla_\mu\psi_\nu+2\bar\Omega^I\gamma^{abc}\psi_a\bar\psi_b\psi_c\\ +ig[\bar\Omega,\,\Omega]^I-\frac{i}{4}\bar\psi_c\gamma^{abcd}\psi_dF_{ab}^I(W)\end{pmatrix}\\
&-\frac{1}{2}\left(\mathcal{N}_{IJ}-\frac{\mathcal{N}_I\mathcal{N}_J}{\mathcal{N}}\right)\left(-\frac{1}{4}F^I(W)\cdot F^J(W)+\frac{1}{2}\nabla_aM^I\nabla^aM^J\right)\\
&-\frac{1}{2}\mathcal{N}_{IJ}\begin{pmatrix}+2i\bar\Omega^I\nabla\!\!\!\!/\,\Omega^J+i\bar\psi_a(\gamma\cdot F(W)-2\nabla\!\!\!\!/ M)^I\gamma^a\Omega^J\\ -2(\bar\Omega^I\gamma^a\gamma^{bc}\psi_a)(\bar\psi_b\gamma_c\Omega^J)+2(\bar\Omega^I\gamma^a\gamma^b\psi_a)(\bar\psi_b\Omega^J)\end{pmatrix}\\
&-\mathcal{N}_{IJK}\begin{pmatrix}-i\bar\Omega^I\frac{1}{4}\gamma\cdot F^J(W)\Omega^K\\ +\frac{2}{3}(\bar\Omega^I\gamma^{ab}\Omega^J)(\bar\psi_a\gamma_b\Omega^K)+\frac{2}{3}(\bar\psi^i\cdot\gamma\Omega^{Ij})(\bar\Omega_{(i}^J\Omega_{j)}^K)\end{pmatrix},\\
e^{-1}\mathcal{L}'_{\rm aux}={}&C'(\mathcal{A}^2+2\mathcal{N})-8i\bar\chi'(\zeta+\Omega)-b^a\nabla_a\mathcal{N}\\
&+2\mathcal{N}\Big(v-\frac{1}{2\alpha}F(A)+\frac{\mathcal{N}_I}{4\mathcal{N}}F^I(W)\Big)^2\\
&-i\Big(v-\frac{1}{2\alpha}F(A)\Big)^{ab}\Big(2\mathcal{N}\bar\psi_a\psi_b+\bar\zeta^{\bar\alpha}\gamma_{ab}\zeta_\alpha-4\bar\psi_a\gamma_b\Omega-\frac{1}{2}\mathcal{N}_{IJ}\bar\Omega^I\gamma_{ab}\Omega^J\Big)\\
&-\mathcal{N}V_a^{ij}V_{ij}^a-V_{ij}^a\Big(2\mathcal{A}^{\bar\alpha i}\nabla_a\mathcal{A}_\alpha^j+4i\bar\Omega^i\psi_a^j-i\mathcal{N}_{IJ}\bar\Omega^{Ii}\gamma_a\Omega^{Jj}\Big)\\
&-\frac{1}{2}\mathcal{N}_{IJ}(Y^{ijI}-M^It^{ij})(Y_{ij}^J-M^Jt_{ij})\\
&+(Y_{ij}^I-M^It_{ij})(2\mathcal{A}_\alpha^i(gt_I)^{\bar\alpha\beta}\mathcal{A}_\beta^j+i\mathcal{N}_{IJK}\bar\Omega^{Ji}\Omega^{Kj})\\
&+\Big(1-A^2/\alpha^2\Big)(\mathcal{F}_i^{\bar\alpha}-\mathcal{F}_{{\rm sol}\,i}^{\bar\alpha})(\mathcal{F}_\alpha^i-\mathcal{F}_{{\rm sol}\,\alpha}^i)\,, \qquad (4\cdot7)
\end{aligned}
$$

where $\mathcal{F}_{{\rm sol}\,i}^\alpha$ is the solution of the equation of motion for $\mathcal{F}_i^\alpha$,

$$
\mathcal{F}_{{\rm sol}\,i}^\alpha=-\frac{1}{2}\alpha m(d^{-1})_\gamma^\alpha\eta^{\gamma\beta}\mathcal{A}_{\beta i}=-\alpha(gt_{I=0})^{\alpha\beta}\mathcal{A}_{\beta i}=(gM^0t_0)^\alpha{}_\beta\mathcal{A}_i^\beta. \qquad (4\cdot8)
$$

Here it is quite remarkable that all the terms explicitly containing either b_μ $(= \alpha^{-1}\partial_\mu\alpha)$ or $F_{\mu\nu}(A)$ have completely disappeared from the action, other than $\mathcal{L}'_{\rm aux}$, except for the terms contained in the form M^I and $F^I(W)$. That is, $\alpha = M^{I=0}$ and $F_{\mu\nu}(A) = F^{I=0}_{\mu\nu}(W)$, which carry the index $I = 0$, do not appear by themselves, but are only contained in the action in a form that is completely symmetric with the components with $I \geq 1$.

§5. Final form of the action

In view of the action (4·7), we note that the Einstein term can be made canonical if

$$\mathcal{N}(M) = 1. \tag{5·1}$$

$\mathcal{N}(M)$ is a cubic function of M^I, but we fortunately have local dilatation $\boldsymbol{D}$ symmetry, so that we can take $\mathcal{N}(M) = 1$ as a gauge fixing condition for the $\boldsymbol{D}$ gauge.[12)]

However, the action (4·7) is still not in the final form, since there remains a mixing kinetic term $4i\mathcal{N}_I\bar{\Omega}^I\gamma^{\mu\nu}\nabla_\mu\psi_\nu$ between the Rarita-Schwinger field ψ^i_μ and the gaugino field component $\Omega_i = \mathcal{N}_I\Omega^I_i$. If we had superconformal symmetry, we could remove the mixing simply by imposing

$$\mathcal{N}_I\Omega^I_i \equiv \Omega_i = 0 \tag{5·2}$$

as a conformal $\boldsymbol{S}$ supersymmetry gauge fixing condition. Unfortunately, we already fixed the $\boldsymbol{S}$ gauge when performing the dimensional reduction from 6D to 5D, and thus we no longer have such $\boldsymbol{S}$ symmetry. Therefore we here must remove the mixing by making field redefinitions. The proper Rarita-Schwinger field is found to be

$$\psi^{\rm N}_{\mu i} = \psi_{\mu i} - \frac{1}{3\mathcal{N}}\gamma_\mu\Omega_i\,. \tag{5·3}$$

We also redefine the gaugino fields as

$$\lambda^I_i \equiv \Omega^I_i - \frac{M^I}{3\mathcal{N}}\Omega_i = \mathcal{P}^I_J\Omega^J_i\,, \tag{5·4}$$

where $\mathcal{P}^I_J$ is the projection operator

$$\mathcal{P}^I_J \equiv \delta^I_J - \frac{M^I\mathcal{N}_J}{3\mathcal{N}} \quad \rightarrow \quad \mathcal{P}^I_J M^J = \mathcal{P}^I_J\mathcal{N}_I = 0\,. \tag{5·5}$$

This new gaugino fields λ^I_i satisfy

$$\mathcal{N}_I\lambda^I_i = 0\,, \tag{5·6}$$

so that they correspond to the gaugino fields Ω^I_i which we would have had if we could have imposed the $\boldsymbol{S}$ gauge fixing condition (5·2). Note, however, that the number of independent

components of λ^I is the same as that of the original Ω^I, since the $I=0$ component of the latter vanishes: $\Omega^{I=0}=0$. Note also that Eq. (5·4) and the relation $\Omega^{I=0}=0$ lead to

$$\lambda_i^0 = -\frac{\alpha}{3\mathcal{N}}\Omega_i, \tag{5·7}$$

so that $\Omega_i = \mathcal{N}^I\Omega_{Ii}$ is now essentially the $I=0$ component of λ_i^I.

We have $\mathcal{A}_i^{\bar{\alpha}}\zeta_a \equiv \zeta_i = -\Omega_i$ on shell, implying that the hypermultiplet fermions ζ_α contain the Ω_i degree of freedom. To separate it out, we define new hypermultiplet fermions ξ_α by

$$\xi_\alpha \equiv \zeta_\alpha - \frac{\mathcal{A}_\alpha^i}{\mathcal{N}}\Omega_i\,. \tag{5·8}$$

Then, ξ_α is indeed orthogonal to $\mathcal{A}_i^{\bar{\alpha}}$ on-shell:

$$\mathcal{A}_i^{\bar{\alpha}}\xi_\alpha = \zeta_i - \frac{\mathcal{A}^2}{2\mathcal{N}}\Omega_i = (\zeta_i+\Omega_i) - \frac{1}{2\mathcal{N}}\Omega_i(\mathcal{A}^2+2\mathcal{N})\,. \tag{5·9}$$

In the Lagrangian, the quadratic terms of the form $\bar{\zeta}^{\bar{\alpha}}\Gamma\zeta_\alpha$ yield 'cross terms' proportional to $\mathcal{A}_i^{\bar{\alpha}}\xi_\alpha$, which do not vanish but can be eliminated by further shifts of the multiplier auxiliary fields χ and C. Explicitly, we have

$$\begin{aligned}
\bar{\zeta}^{\bar{\alpha}}\nabla\!\!\!\!/\,\zeta_\alpha &= (\bar{\zeta}^{\bar{\alpha}}\nabla\!\!\!\!/\,\zeta_\alpha)' + \left\{\frac{1}{\mathcal{N}}\Big(e^{-1}\nabla_\mu(ee_a^\mu)\,\bar{\Omega}^i + 2\nabla_a\bar{\Omega}^i\Big)\gamma^a(\zeta_i+\Omega_i)\right.\\
&\qquad\qquad\left. + \frac{1}{2\mathcal{N}^2}\bar{\Omega}\nabla\!\!\!\!/\,\Omega(\mathcal{A}^2+2\mathcal{N})\right\},\\
\bar{\zeta}^{\bar{\alpha}}\Gamma\zeta_\alpha &= (\bar{\zeta}^{\bar{\alpha}}\Gamma\zeta_\alpha)' - \left\{\frac{2}{\mathcal{N}}\bar{\Omega}^i\Gamma(\zeta_i+\Omega_i) - \frac{1}{2\mathcal{N}^2}\bar{\Omega}\Gamma\Omega(\mathcal{A}^2+2\mathcal{N})\right\},
\end{aligned} \tag{5·10}$$

up to a total derivative term in the action, where the primed terms are the 'diagonal' parts:

$$\begin{aligned}
(\bar{\zeta}^{\bar{\alpha}}\nabla\!\!\!\!/\,\zeta_\alpha)' &\equiv \bar{\xi}^{\bar{\alpha}}\nabla\!\!\!\!/\,\xi_\alpha + \frac{1}{\mathcal{N}}\bar{\Omega}\nabla\!\!\!\!/\,\Omega + \frac{1}{\mathcal{N}^2}(\bar{\Omega}^i\gamma^a\Omega^j)\mathcal{A}_i^{\bar{\alpha}}\nabla_a\mathcal{A}_{\alpha j} + \frac{2}{\mathcal{N}}(\bar{\xi}^{\bar{\alpha}}\gamma^a\Omega_i)\nabla_a\mathcal{A}_\alpha^i\,,\\
(\bar{\zeta}^{\bar{\alpha}}\gamma_{ab}\zeta_\alpha)' &\equiv \bar{\xi}^{\bar{\alpha}}\gamma_{ab}\xi_\alpha + \frac{1}{\mathcal{N}}\bar{\Omega}\gamma_{ab}\Omega\,.
\end{aligned} \tag{5·11}$$

Collecting all the contributions from the bilinear terms in ζ_α, we find that the cross terms can be eliminated by replacing C' and χ' by the shifted quantities C'' and χ'' defined as

$$\begin{aligned}
C'' &= C' + \frac{1}{2\mathcal{N}^2}\left\{-2i\bar{\Omega}\nabla\!\!\!\!/\,\Omega + \Big(\bar{\psi}_a\gamma_b\psi_c(\bar{\Omega}\gamma^{abc}\Omega) - \tfrac{1}{2}\bar{\psi}^a\gamma_{bc}\psi_a(\bar{\Omega}\gamma^{bc}\Omega)\Big)\right.\\
&\qquad\qquad\left. - i\bar{\Omega}\gamma\cdot(v-\tfrac{1}{2\alpha}F(A))\Omega\right\} + \frac{i}{\mathcal{N}}e^{-1}\nabla_\mu(e\bar{\psi}_a\gamma^\mu\gamma^a\Omega)\,,\\
\chi_i'' &= \chi_i' + \frac{1}{4\mathcal{N}}\left\{\Big(e^{-1}\nabla_\mu(ee_a^\mu)\,\gamma^a\Omega_i + 2\nabla\!\!\!\!/\,\Omega_i\Big)\right.\\
&\qquad \left. + i\,\Big(\gamma^{abc}\Omega_i(\bar{\psi}_a\gamma_b\psi_c) - \tfrac{1}{2}\gamma^{bc}\Omega_i(\bar{\psi}^a\gamma_{bc}\psi_a)\Big) + \gamma\cdot(v-\tfrac{1}{2\alpha}F(A))\Omega_i\right\}.
\end{aligned} \tag{5·12}$$

Here, in the last term of C'', we have also added a contribution from the term $-4i\bar{\psi}^i_a\gamma^b\gamma^a\zeta_\alpha\nabla_b\mathcal{A}^{\bar{\alpha}}_i$ in $\mathcal{L}'_{\text{hyper}}$, which yields a term proportional to $\mathcal{A}^2+2\mathcal{N}$ after partial integration when ζ_α is rewritten by using Eq. (5·8).

We now rewrite the action (4·7) by using the field redefinitions (5·3), (5·4) and (5·8) everywhere. From this point, *the Rarita-Schwinger field always stands for the new variable* ψ^{N}_μ*, and we omit the cumbersome superscript* N.

Rewriting (4·7) actually involves a very tedious computation. Note, for instance, that the spin connection $\omega^{ab}_\mu|_{b_\mu=0}$ contained in the covariant derivative ∇_μ and $\mathcal{R}(M)$ is given in Eq. (4·4) in terms of the original Rarita-Schwinger field ψ_μ, which should also be rewritten in terms of the new variable ψ^{N}_μ in Eq. (5·3). Surprisingly, however, all the terms containing $\varOmega_i \equiv \mathcal{N}^I\varOmega_{Ii}$ completely cancel out in the action if the auxiliary fields are eliminated by the equations of motion. This action, which is obtained by eliminating the auxiliary fields, is just the action in the on-shell formulation, which we term the 'on-shell action'. Since $\varOmega_i \propto \lambda^{I=0}_i$, as noted above, this fact that the $\varOmega_i$ completely disappear is the *fermionic counterpart* of the previously observed fact that the $M^{I=0}=\alpha$ and $F^{I=0}_{\mu\nu}(W)=F_{\mu\nu}(A)$ terms disappeared from the action. That is, there appear no terms that carry an explicit $I=0$ index, and the upper indices I, J, etc., are always contracted with the lower indices of $\mathcal{N}_I, \mathcal{N}_{IJ}$, etc., in the on-shell action.

We can demonstrate this noteworthy fact as follows. First, we can confirm that the index I is 'conserved' in all the supersymmetry transformation laws of the physical fields (fields other than the auxiliary fields); that is, the supersymmetry transformation of a physical field with the index I contains only the terms carrying the same index, and that of a physical field without the index I contains only the terms carrying no index. Thus the fields $\varOmega_i$, α and $F_{\mu\nu}(A)$, carrying the $I=0$ index explicitly, appear only in the transformation of those $I=0$ fields. This can be confirmed relatively easily, as we see in the next section. Therefore, if such terms carrying the $I=0$ index explicitly remain in the on-shell action, the supersymmetry invariance of the action implies that the parts of the action containing different numbers of $I=0$ fields are separately supersymmetry invariant. But we know already that the bosonic $I=0$ fields α and $F_{\mu\nu}(A)$ do not appear. Clearly, no such invariant term can be made from the $\varOmega_i$ without using their superpartners α and $F_{\mu\nu}(A)$. This proves the total cancellation of the $\varOmega_i$ terms in the on-shell action. (We have also confirmed this cancellation explicitly by direct rewriting of the action, except for some four-fermion term parts.)

Completing the square of the auxiliary field terms in the action (4·7), we can rewrite the action in a sum of the on-shell action and the perfect square terms of the auxiliary fields. The auxiliary fields implicitly contain $\varOmega_i$-dependent terms in them. This can be seen by substituting the field redefinitions (5·3), (5·4) and (5·8) into their solutions of the equations

of motion. If we redefine the auxiliary fields as follows by subtracting these implicitly $\varOmega_i$-dependent terms, then the $\varOmega_i$-dependent terms completely disappear also from the perfect square terms of the auxiliary fields, and we have

$$
\begin{aligned}
\tilde{V}_a^{ij} &= V_a^{ij} + \frac{1}{2\mathcal{N}}\Big(4i\bar{\varOmega}^{(i}\psi_a^{j)} + \frac{2i}{3\mathcal{N}}\bar{\varOmega}^i\gamma_a\varOmega^j\Big), \\
\tilde{v}_{ab} &= v_{ab} - \frac{1}{2\alpha}F_{ab}(A) + i\bar{\psi}_a\psi_b + i\frac{2}{3\mathcal{N}}\bar{\psi}_{[a}\gamma_{b]}\varOmega + \frac{i}{9\mathcal{N}^2}\bar{\varOmega}\gamma_{ab}\varOmega, \\
\tilde{Y}^{Iij} &= \mathcal{P}^I_J Y^{Jij} - \frac{2i}{3\mathcal{N}}\bar{\lambda}^{I(i}\varOmega^{j)}\,, \\
\tilde{t}^{ij} &= t^{ij} - \frac{\mathcal{N}_I Y^{Iij}}{3\mathcal{N}} + \frac{i}{9\mathcal{N}^2}\bar{\varOmega}^{(i}\varOmega^{j)},
\end{aligned}
\qquad (5{\cdot}13)
$$

where $\mathcal{P}^I_J$ is the projection operator introduced in Eq. (5·5), and we have taken into account the fact that $Y^I - M^I t = \mathcal{P}^I_J Y^J - M^I(t - \mathcal{N}_J Y^J/3\mathcal{N})$. Note that the vector multiplet auxiliary fields $\tilde{Y}^I$ as well as $\mathcal{P}^I_J Y^J$ are orthogonal to $\mathcal{N}_I$, as are the fermionic partners λ^I. The solutions of the equations of motion for these auxiliary fields are now free from the $\varOmega_i$ and given by

$$
\begin{aligned}
\tilde{V}^{ij}_{\mathrm{sol}\,a} &= -\frac{1}{2\mathcal{N}}\Big(2\mathcal{A}^{\bar{\alpha}(i}\nabla_a\mathcal{A}^{j)}_\alpha - i\mathcal{N}_{IJ}\bar{\lambda}^{Ii}\gamma_a\lambda^{Jj}\Big), \\
\tilde{v}_{\mathrm{sol}\,ab} &= -\frac{1}{4\mathcal{N}}\Big\{\mathcal{N}_I F_{ab}(W)^I - i\left(6\mathcal{N}\bar{\psi}_a\psi_b + \bar{\xi}^{\bar{\alpha}}\gamma_{ab}\xi_\alpha - \tfrac{1}{2}\mathcal{N}_{IJ}\bar{\lambda}^I\gamma_{ab}\lambda^J\right)\Big\}, \\
\tilde{Y}^{Iij}_{\mathrm{sol}} &= -\tfrac{1}{2}a^{IJ}\mathcal{P}^K_J\mathcal{Y}^{ij}_K = -\tfrac{1}{2}\mathcal{P}^I_J a^{JK}\mathcal{Y}^{ij}_K = -\left(\tfrac{1}{2}a^{IJ} - \tfrac{1}{3}M^I M^J\right)\mathcal{Y}^{ij}_J \\
&\qquad \text{with} \quad \mathcal{Y}^{ij}_I \equiv 2\mathcal{A}^{(i}_\alpha(gt_I)^{\bar{\alpha}\beta}\mathcal{A}^{j)}_\beta + i\mathcal{N}_{IJK}\bar{\lambda}^{Ji}\lambda^{Kj}, \\
\tilde{t}^{ij}_{\mathrm{sol}} &= -\frac{1}{6\mathcal{N}}M^I\mathcal{Y}^{ij}_I = -\frac{1}{6\mathcal{N}}\Big(2\mathcal{A}^{(i}_\alpha(gM)^{\bar{\alpha}\beta}\mathcal{A}^{j)}_\beta + i\mathcal{N}_{IJ}\bar{\lambda}^{Ii}\lambda^{Jj}\Big),
\end{aligned}
\qquad (5{\cdot}14)
$$

where a^{IJ} is the inverse of the metric a_{IJ} of the vector multiplet kinetic terms:

$$
a_{IJ} \equiv -\tfrac{1}{2}\frac{\partial^2}{\partial M^I\partial M^J}\ln\mathcal{N} = -\frac{1}{2\mathcal{N}}\Big(\mathcal{N}_{IJ} - \frac{\mathcal{N}_I\mathcal{N}_J}{\mathcal{N}}\Big), \quad a^{IJ} \equiv (a^{-1})^{IJ}. \qquad (5{\cdot}15)
$$

It possesses the properties

$$
a_{IJ}M^J = \mathcal{N}_I/2\mathcal{N} \;\rightarrow\; a^{IJ}\mathcal{N}_J/2\mathcal{N} = M^I, \qquad a^{IJ}\mathcal{P}^K_J = \mathcal{P}^I_J a^{JK}. \qquad (5{\cdot}16)
$$

We here have assumed that a_{IJ} is invertible. However, there are some interesting cases in which $\det(a_{IJ}) = 0$. Such a situation implies that some vector multiplets have no kinetic terms, since a_{IJ} gives the metric of the vector multiplets. We comment on such a possibility below.

After all of the above calculations, the action is finally found to take the form

$$
\mathcal{L} = \mathcal{L}_{\mathrm{hyper}} + \mathcal{L}_{\mathrm{vector}} + \mathcal{L}_{\mathrm{C\text{-}S}} + \mathcal{L}_{\mathrm{aux}}\ ,
$$

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm hyper} &= \nabla^a \mathcal{A}_i^{\bar\alpha}\nabla_a\mathcal{A}_\alpha^i - 2i\bar\xi^{\bar\alpha}(\nabla\!\!\!\!/ + gM)\xi_\alpha \\
&\quad + \mathcal{A}_i^{\bar\alpha}(gM)^2{}_\alpha{}^\beta\mathcal{A}_\beta^i - 4i\bar\psi_a^i\gamma^b\gamma^a\xi_\alpha\nabla_b\mathcal{A}_i^{\bar\alpha} - 2i\bar\psi_a^{(i}\gamma^{abc}\psi_c^{j)}\mathcal{A}_j^{\bar\alpha}\nabla_b\mathcal{A}_{\alpha i} \\
&\quad + \mathcal{A}_i^{\bar\alpha}\Big(8ig\bar\lambda_{\alpha\beta}^i\xi^\beta - 4ig\bar\psi_a^i\gamma^a M_{\alpha\beta}\xi^\beta \\
&\qquad\qquad + 4ig\bar\psi_a^{(i}\gamma^a\lambda_{\alpha\beta}^{j)}\mathcal{A}_j^\beta - 2ig\bar\psi_a^{(i}\gamma^{ab}\psi_b^{j)}M_{\alpha\beta}\mathcal{A}_j^\beta\Big) \\
&\quad + \bar\psi_a\gamma_b\psi_c\bar\xi^{\bar\alpha}\gamma^{abc}\xi_\alpha - \frac{1}{2}\bar\psi^a\gamma^{bc}\psi_a\bar\xi^{\bar\alpha}\gamma_{bc}\xi_\alpha\ , \\
e^{-1}\mathcal{L}_{\rm vector} &= -\frac{1}{2}R(\omega) - 2i\bar\psi_\mu\gamma^{\mu\nu\rho}\nabla_\nu\psi_\rho + (\bar\psi_a\psi_b)(\bar\psi_c\gamma^{abcd}\psi_d + \bar\psi^a\psi^b) \\
&\quad - \mathcal{N}_I\left(ig[\bar\lambda,\,\lambda]^I - \frac{i}{4}\bar\psi_c\gamma^{abcd}\psi_d F_{ab}(W)^I\right) \\
&\quad + a_{IJ}\left(\begin{array}{c} -\frac{1}{4}F(W)^I\cdot F(W)^J + \frac{1}{2}\nabla_a M^I\nabla^a M^J \\ +2i\bar\lambda^I\nabla\!\!\!\!/\lambda^J + i\bar\psi_a(\gamma\cdot F(W) - 2\nabla\!\!\!\!/ M)^I\gamma^a\lambda^J \\ -2(\bar\lambda^I\gamma^a\gamma^{bc}\psi_a)(\bar\psi_b\gamma_c\lambda^J) + 2(\bar\lambda^I\gamma^a\gamma^b\psi_a)(\bar\psi_b\lambda^J) \end{array}\right) \\
&\quad - \mathcal{N}_{IJK}\left(\begin{array}{c} -i\bar\lambda^I\frac{1}{4}\gamma\cdot F(W)^J\lambda^K \\ +\frac{2}{3}(\bar\lambda^I\gamma^{ab}\lambda^J)(\bar\psi_a\gamma_b\lambda^K) + \frac{2}{3}(\bar\psi^i\cdot\gamma\lambda^{Ij})(\bar\lambda^J_{(i}\lambda^K_{j)}) \end{array}\right) \\
&\quad +\frac{1}{8}\left(2\bar\psi_a\psi_b + \bar\xi^{\bar\alpha}\gamma_{ab}\xi_\alpha + a_{IJ}\bar\lambda^I\gamma_{ab}\lambda^J\right)^2 \\
&\quad + i\frac{1}{4}\mathcal{N}_I F(W)^I\left(2\bar\psi_a\psi_b + \bar\xi^{\bar\alpha}\gamma_{ab}\xi_\alpha + a_{IJ}\bar\lambda^I\gamma_{ab}\lambda^J\right) \\
&\quad + \left(\mathcal{A}^{\bar\alpha i}\nabla_a\mathcal{A}_\alpha^j + ia_{IJ}\bar\lambda^{Ii}\gamma_a\lambda^{Jj}\right)^2 \\
&\quad - \frac{1}{4}(a^{IJ} - M^IM^J)\mathcal{Y}_I^{ij}\mathcal{Y}_{J\,ij}\ .
\end{aligned}
\tag{5·17}
$$

Here $\mathcal{L}_{\rm aux}$ represents the perfect square terms of the auxiliary fields, which vanish on shell:

$$
\begin{aligned}
e^{-1}\mathcal{L}_{\rm aux} &= C'''(\mathcal{A}^2 + 2) - 8i\bar\chi''^i\mathcal{A}_i^{\bar\alpha}\xi_\alpha \\
&\quad + 2(\tilde v - \tilde v_{\rm sol})^2 - (\tilde V - \tilde V_{\rm sol})^{ij}(\tilde V - \tilde V_{\rm sol})_{ij} \\
&\quad - 3(\tilde t - \tilde t_{\rm sol})^{ij}(\tilde t - \tilde t_{\rm sol})_{ij} + a_{IJ}(\tilde Y^I - \tilde Y^I_{\rm sol})^{ij}(\tilde Y^J - \tilde Y^J_{\rm sol})_{ij} \\
&\quad + \left(1 - A^2/\alpha^2\right)(\mathcal{F}_i^{\bar\alpha} - \mathcal{F}^{\bar\alpha}_{{\rm sol}\,i})(\mathcal{F}_\alpha^i - \mathcal{F}^i_{{\rm sol}\,\alpha})\ .
\end{aligned}
\tag{5·18}
$$

Here the multiplier term $C''(\mathcal{A}^2 + 2\mathcal{N}) - 8i\bar\chi''(\zeta + \Omega)$ has been rewritten into the form of the first line by using Eq. (5·9) and defining C''' in terms of the C'' field as

$$
C''' = C'' - i\frac{4}{\mathcal{N}}\bar\chi''\Omega\,. \tag{5·19}
$$

Expressed in this way, the explicit Ω_i have been completely removed from the action. Note that the final action (5·17) with (5·18) is everywhere written in terms of the new variables, although the superscript N has been omitted. In particular, the spin connection ω_μ^{ab} in the

covariant derivative ∇_μ and $R(\omega)$ is the new one given by Eq. (4·4) with the new ψ_μ used and b_μ set equal to 0. By using this ω_μ^{ab}, $R(\omega)$ is given as usual:

$$R_{\mu\nu}{}^{ab}(\omega) = 2\partial_{[\mu}\omega_{\nu]}{}^{ab} - 2\omega_{[\mu}{}^{[ac}\omega_{\nu]c}{}^{b]}, \quad R_{ab}(\omega) \equiv R_{ac}{}^{c}{}_{b}(\omega)\,, \quad R(\omega) \equiv R_a{}^a(\omega)\,. \qquad (5·20)$$

§6. Supersymmetry transformation

Now we should modify the supersymmetry ($\boldsymbol{Q}$) transformation $\delta_Q(\varepsilon)$, since we have fixed the $\boldsymbol{D}$ gauge by (5·1) and made various field redefinitions, (5·3), (5·4) and (5·8). The proper $\boldsymbol{Q}$ transformation is found to be given by the following linear combination of the original transformations of $\boldsymbol{Q}$, dilatation $\boldsymbol{D}$, local-Lorentz $\boldsymbol{M}$ and $SU(2)$ $\boldsymbol{U}$:

$$\begin{aligned}\delta_Q^{\rm N}(\varepsilon) &= \delta_Q(\varepsilon) + \delta_D(\rho(\varepsilon)) + \delta_M(\lambda^{ab}(\varepsilon)) + \delta_U(\theta^{ij}(\varepsilon)),\\ \rho(\varepsilon) &\equiv -\frac{2i}{3\mathcal{N}}\bar\varepsilon\Omega, \quad \lambda^{ab}(\varepsilon) \equiv \frac{2i}{3\mathcal{N}}\bar\varepsilon\gamma^{ab}\Omega, \quad \theta^{ij}(\varepsilon) \equiv -\frac{2i}{\mathcal{N}}\bar\varepsilon^{(i}\Omega^{j)}.\end{aligned} \qquad (6·1)$$

The dilatation part $\delta_D(\rho(\varepsilon))$ is determined so as to maintain the $\boldsymbol{D}$ gauge fixing condition (5·1): $(\delta_Q(\varepsilon) + \delta_D(\rho(\varepsilon)))\mathcal{N} = 0$. The local-Lorentz part $\delta_M(\lambda^{ab}(\varepsilon))$ is fixed by requiring that the transformation of the fünfbein take the canonical form $\delta^{\rm N}(\varepsilon)e_\mu{}^a = -2i\bar\varepsilon\gamma^a\psi_\mu^{\rm N}$ in terms of the *new* Rarita-Schwinger field $\psi_\mu^{\rm N}$. In the first part of this section, we revive the superscript N to distinguish the new variables from the original ones. Finally, the $SU(2)$ part $\delta_U(\theta^{ij})$ is added so that the hypermultiplet scalar field $\mathcal{A}_\alpha^i$ is transformed in the new fermion component ξ_α to yield the form $\delta^{\rm N}(\varepsilon)\mathcal{A}_\alpha^i = 2i\bar\varepsilon^i\xi_\alpha$.

To write the supersymmetry transformation rules concisely and covariantly, we should use the supercovariant derivative $\hat{\mathcal{D}}_\mu$ and the supercovariantized curvatures $\hat{\mathcal{R}}_{\mu\nu}$. But these supercovariant quantities are also modified by the $\boldsymbol{D}$ gauge fixing and field redefinitions. We define a new supercovariant derivative $\hat{\mathcal{D}}_\mu^{\rm N}$ in the usual form, but by using the new gauge fields and the new supersymmetry transformation:

$$\hat{\mathcal{D}}_\mu^{\rm N} = \partial_\mu - \delta_M(\omega_\mu^{{\rm N}\,ab}) - \delta_U(\tilde V_\mu^{ij}) - \delta_G(W_\mu) - \delta_Q^{\rm N}(\psi_\mu^{\rm N}). \qquad (6·2)$$

The relation with the original supercovariant derivative $\hat{\mathcal{D}}_\mu$, which also contains the $\boldsymbol{D}$ covariantization, is found to be given by

$$\hat{\mathcal{D}}_\mu = \hat{\mathcal{D}}_\mu^{\rm N} - \delta_D(b_\mu^{\rm N}) + \delta_M\left(2e_\mu{}^{[a}b^{{\rm N}b]} + \frac{i}{9\mathcal{N}^2}\bar\Omega\gamma_\mu{}^{ab}\Omega\right) - \delta_U\left(\frac{i}{3\mathcal{N}^2}\bar\Omega^{(i}\gamma_\mu\Omega^{j)}\right) - \delta_Q^{\rm N}\left(\frac{1}{3\mathcal{N}}\gamma_\mu\Omega\right), \quad (6·3)$$

where $b_\mu^{\rm N}$ is the supercovariantized b_μ $(= \alpha^{-1}\partial_\mu\alpha)$ defined as $b_\mu^{\rm N} \equiv \alpha^{-1}\hat{\mathcal{D}}_\mu^{\rm N}\alpha = b_\mu + \frac{2i}{3\alpha\mathcal{N}}\bar\psi_\mu^{\rm N}\Omega$. The new curvatures $\hat{\mathcal{R}}_{ab}^{{\rm N}\,\bar A}$ are defined as usual by Eq. (I 2·28) by using the new covariant derivative $\hat{\mathcal{D}}_a^{\rm N}$ with flat index a: $[\hat{\mathcal{D}}_a^{\rm N}, \hat{\mathcal{D}}_b^{\rm N}] = -\hat{\mathcal{R}}_{ab}^{{\rm N}\,\bar A}\boldsymbol{X}_{\bar A}$. Hence, using the relation (6·3)

between $\hat{\mathcal{D}}_\mu$ and $\hat{\mathcal{D}}_\mu^{\mathrm{N}}$, we can find the relations between the new curvatures and the original curvatures $\hat{\mathcal{R}}_{ab}{}^{\bar{A}}$. The Yang-Mills group G is also regarded as a subgroup of our supergroup, and so, for example, in the case $\bar{A} = I$ of G, we find

$$\hat{F}^I_{ab}(W) = \hat{F}^{\mathrm{N}I}_{ab}(W) + \tfrac{4i}{3\mathcal{N}}\bar{\Omega}\gamma_{ab}\lambda^I + \tfrac{2i}{9\mathcal{N}^2}M^I\bar{\Omega}\gamma_{ab}\Omega\,. \qquad (6{\cdot}4)$$

From this point, we again suppress the cumbersome superscript N of ψ^{N}_μ, $\omega^{\mathrm{N}\,ab}_\mu$, $\hat{\mathcal{D}}^{\mathrm{N}}_\mu$, $\hat{\mathcal{R}}^{\mathrm{N}\ A}_{\mu\nu}$ ($\hat{F}^{\mathrm{N}I}_{\mu\nu}$) and $\delta^{\mathrm{N}}_Q(\varepsilon)$, since *every quantity that appears in the following is always one of these new ones.*

As mentioned in the preceding section, we find that the (new) supersymmetry transformation 'conserves' the index I, and thus the $\Omega_i \propto \lambda^{I=0}_i$, as well as $F_{ab}(A) = F^{I=0}_{ab}(W)$ and $\alpha = M^{I=0}$ (or $b_\mu = \alpha^{-1}\partial_\mu\alpha$), carrying an $I = 0$ index, do not explicitly appear in the transformation laws, unless the transformed field itself carries $I = 0$. (The only exception is the transformation $\delta\mathcal{F}^i_\alpha$, which contains $A_\mu = W^{I=0}_\mu$ and $\alpha = M^{I=0}$ explicitly. However, $\mathcal{F}^i_\alpha$ is defined to be $\delta_Z(\alpha)\mathcal{A}^i_\alpha$ with $\alpha = M^{I=0}$, and so it may be regarded as carrying the index $I = 0$ implicitly.) It is quite easy to demonstrate the disappearance of $F_{ab}(A) = F^{I=0}_{ab}(W)$ and $\alpha = M^{I=0}$ by direct computation.

To see the disappearance of explicit Ω_i factors, however, we have proceeded in the following way. For the physical fields, $e_\mu{}^a, \psi^i_\mu, W^I_\mu, M^I, \lambda^I, \mathcal{A}^i_\alpha, \xi_\alpha$, we have explicitly computed their supersymmetry transformation laws and directly checked that the explicit Ω_i cancel out completely. For the auxiliary fields $\phi = \tilde{V}^{ij}_\mu, \tilde{t}^{ij}, \tilde{v}_{ab}, \tilde{Y}^{ij}, \mathcal{F}^i_\alpha$, other than χ'' and C''', such rigorous computations become quite tedious, and so we checked the cancellation indirectly: For such auxiliary fields ϕ, the supersymmetry transformation of $\phi - \phi_{\mathrm{sol}}$, $\delta(\phi - \phi_{\mathrm{sol}})$, should vanish on-shell, that is, when the equations of motion for auxiliary fields are used. (But note that the equations of motion for the physical fields need not be used.) Therefore, if an Ω_i appears explicitly in $\delta(\phi - \phi_{\mathrm{sol}})$, it must be multiplied by the factors $(\phi - \phi_{\mathrm{sol}})$ which vanish on-shell, or it must appear in the form $\Omega_i + \zeta_i$. For the former possibility, we can easily see whether such terms appear or not, by keeping track of auxiliary fields explicitly. It is seen that the latter possibility does not occur by confirming that $\zeta_i = \mathcal{A}^{\bar{\alpha}}_i\zeta_\alpha$ never appears in $\delta(\phi - \phi_{\mathrm{sol}})$. Once Ω_i is seen to be absent in $\delta(\phi - \phi_{\mathrm{sol}})$, it is seen that it does not appear in $\delta\phi$ either, since ϕ_{sol} consists of physical fields alone, and hence $\delta\phi_{\mathrm{sol}}$ does not contain any Ω_i explicitly.

Computations to derive transformation laws of the new auxiliary fields χ'' and C''' directly from those of the original fields χ and C become terribly tedious, because the relations between these new and original fields are very complicated. Instead of doing this, we can use the invariance of the action to find $\delta\chi''$ and $\delta C'''$. Then, since they appear in the form $\delta C'''\,(\mathcal{A}^2{+}2\mathcal{N}){-}8i\delta\bar{\chi}''^i\,\mathcal{A}^{\bar{\alpha}}_i\xi_\alpha$ in $\delta\mathcal{L}$, we have only to compute the terms whose supersymmetry

transformations can yield the factor $\mathcal{A}^2$ or $\mathcal{A}_i^{\bar\alpha}\xi_\alpha$. There are not a great number of such terms in the action. The cancellation condition for the terms proportional to $(\mathcal{A}^2+2\mathcal{N})$ and $\mathcal{A}_i^{\bar\alpha}\xi_\alpha$ determines the supersymmetry transformation laws $\delta C'''$ and $\delta\chi''$ as follows:

$$\begin{aligned}
\delta\chi''^i &= \frac{1}{2}\varepsilon^i C''' + \chi''^i(2i\bar\varepsilon\gamma\cdot\psi) - \psi_a^j(2i\bar\varepsilon_j\gamma^a\chi''^i) - \frac{1}{2}\nabla\!\!\!\!/(\boldsymbol{\Gamma}\varepsilon^i) \\
&\quad + \frac{i}{2}\gamma^a\boldsymbol{\Gamma}\varepsilon^i(\bar\psi_a\gamma\cdot\psi) - \frac{i}{4}\gamma^{abc}\boldsymbol{\Gamma}\varepsilon^i(\bar\psi_a\gamma_b\psi_c) + \frac{i}{8}\gamma^{ab}\boldsymbol{\Gamma}\varepsilon^i(\bar\psi^c\gamma_{ab}\psi_c + 2\bar\psi_a\psi_b) \\
&\quad - \frac{1}{4}\gamma\cdot\tilde v\,\boldsymbol{\Gamma}\varepsilon^i - \frac{1}{2}e^{-1}\nabla_\lambda(e\tilde V^\lambda)\varepsilon^i + \frac{i}{2}e^{-1}\nabla_\lambda(e\bar\psi_\mu^i\gamma^{\mu\lambda\nu}\psi_\nu^j)\varepsilon_j\,, \\
\delta C''' &= -2ie^{-1}\nabla_\mu(e\bar\varepsilon\gamma^\mu\chi'') - ie^{-1}\nabla_\mu(e\bar\psi_a\gamma^\mu\gamma^a\boldsymbol{\Gamma}\varepsilon) \\
&\quad + C'''2i\bar\varepsilon\gamma\cdot\psi + 4i\bar\chi''\boldsymbol{\Gamma}\varepsilon\,.
\end{aligned} \tag{6·5}$$

Here $\boldsymbol{\Gamma}$ is a field-dependent matrix acting on a spinor with an $SU(2)$ index which is defined by

$$\boldsymbol{\Gamma}\varepsilon^i \equiv (-\gamma\cdot\tilde V + 3\tilde t)^i{}_j\varepsilon^j + \gamma\cdot\tilde v\varepsilon^i + \frac{\mathcal{N}_I}{4\mathcal{N}}\gamma\cdot\hat F^I(W)\varepsilon^i + \frac{\mathcal{N}_{IJ}}{\mathcal{N}}\lambda^{iI}(2i\bar\lambda^J\varepsilon)\,. \tag{6·6}$$

Note that there appear derivative terms of the transformation parameter, $\partial_\mu\varepsilon^i$, in these, implying that χ'' and C''' are not covariant quantities. For this reason we redefine these fields once again as follows by adding proper supercovariantization terms:

$$\tilde\chi^i \equiv \chi''^i + \frac{1}{2}\gamma^a\boldsymbol{\Gamma}\psi_a^i\,, \qquad \tilde C \equiv C''' + 2i\bar\psi\cdot\gamma\tilde\chi - i\bar\psi_a\gamma^{ab}\boldsymbol{\Gamma}\psi_b\,. \tag{6·7}$$

Here we have used the identity $(\nabla_\mu\bar\varepsilon)\boldsymbol{\Gamma}\psi_a = \bar\psi_a\boldsymbol{\Gamma}\nabla_\mu\varepsilon$ in deriving the covariantization terms for C'''.

We must next derive the supersymmetry transformation law for these covariant variables $\tilde\chi$ and $\tilde C$ from Eq. (6·5). Note here the simple fact that the transformation of any covariant quantity gives a covariant quantity and hence cannot contain gauge fields explicitly; that is, gauge fields can appear only implicitly in the covariant derivatives or in the form of supercovariant curvatures (field strengths). Otherwise, the two sides of the commutation relation of the transformations would lead to a contradiction. This observation greatly simplifies the computations of $\delta\tilde\chi$ and $\delta\tilde C$, in which we can discard such explicit gauge field terms, since they are guaranteed to cancel out anyway.

We now write the final supersymmetry transformation laws derived this way. The $\boldsymbol{Q}$ transformation laws of the Weyl multiplet are

$$\begin{aligned}
\delta e_\mu{}^a &= -2i\bar\varepsilon\gamma^a\psi_\mu\,, \\
\delta\psi_\mu^i &= \mathcal{D}_\mu\varepsilon^i + \gamma_\mu\tilde t^i{}_j\varepsilon^j + \frac{1}{2}\gamma_{\mu ab}\varepsilon^i\tilde v^{ab} + \frac{\mathcal{N}_I}{12\mathcal{N}}\gamma_\mu\gamma\cdot\hat F^I(W)\varepsilon^i + \frac{\mathcal{N}_{IJ}}{3\mathcal{N}}\gamma_\mu\lambda^{Ii}(2i\bar\lambda^J\varepsilon)\,, \\
\delta\tilde V_\mu^{ij} &= -4i\bar\varepsilon^{(i}\gamma_\mu\tilde\chi^{j)} - i\bar\varepsilon^{(i}\gamma_{\mu ab}\hat{\mathcal{R}}^{abj)}(Q) + 4i\bar\varepsilon^{(i}\gamma\cdot\left(\tilde v + \frac{\mathcal{N}_I}{4\mathcal{N}}\hat F^I(W)\right)\psi_\mu^{j)} \\
&\quad - 6i(\bar\varepsilon\psi_\mu)\tilde t^{ij} + \frac{4\mathcal{N}_{IJ}}{\mathcal{N}}\left((\bar\psi_\mu\lambda^I)\bar\varepsilon^{(i}\lambda^{j)J} - (\bar\varepsilon\lambda^I)\bar\psi_\mu^{(i}\lambda^{j)J}\right),
\end{aligned}$$

$$
\begin{aligned}
\delta\tilde{t}^{ij} &= 4i\bar{\varepsilon}^{(i}\tilde{\chi}^{j)} + i\bar{\varepsilon}^{(i}\gamma\cdot\hat{\mathcal{R}}^{j)}(Q) + \tfrac{2i\mathcal{N}_{IJ}}{3\mathcal{N}}\Big(\bar{\varepsilon}^{(i}\hat{\not{\mathcal{D}}}M^I\lambda^{j)J} - (\bar{\varepsilon}\lambda^I)\tilde{Y}^{Jij}\Big)\,,\\
\delta\tilde{v}_{ab} &= -2i\bar{\varepsilon}\gamma_{ab}\tilde{\chi} - \tfrac{i}{2}\bar{\varepsilon}\gamma_{ab}\gamma\cdot\hat{\mathcal{R}}(Q) + \tfrac{i}{4}\bar{\varepsilon}\gamma_{abcd}\hat{\mathcal{R}}^{cd}(Q) - i\bar{\varepsilon}\hat{\mathcal{R}}_{ab}(Q)\,,\\
\delta\tilde{\chi}^i &= \tfrac{1}{2}\varepsilon^i\tilde{C} - \tfrac{1}{2}(\hat{\not{\mathcal{D}}}\boldsymbol{\Gamma}')\varepsilon^i - \tfrac{1}{4}\gamma\cdot\tilde{v}\,\boldsymbol{\Gamma}'\varepsilon^i + \tfrac{1}{4}\gamma\cdot\hat{\mathcal{R}}(U)\varepsilon^i\\
&\quad+\tfrac{1}{2}\gamma^a\boldsymbol{\Gamma}'\Big(\gamma_a\tilde{t}\varepsilon^i + \tfrac{1}{2}\gamma_{abc}\varepsilon^i\tilde{v}^{bc} + \tfrac{\mathcal{N}_I}{12\mathcal{N}}\gamma_a\gamma\cdot\hat{F}^I(W)\varepsilon^i - \tfrac{\mathcal{N}_{IJ}}{3\mathcal{N}}\gamma_a\lambda^{Ii}(2i\bar{\varepsilon}\lambda^J)\Big)\,,\\
\delta\tilde{C} &= -2i\bar{\varepsilon}\hat{\not{\mathcal{D}}}\tilde{\chi} + \tfrac{1}{2}i\bar{\varepsilon}\{\gamma^{ab},\,\boldsymbol{\Gamma}'\}\hat{\mathcal{R}}_{ab}(Q) + \tfrac{2i}{3}\bar{\varepsilon}\boldsymbol{\Gamma}'\tilde{\chi} + \tfrac{i}{3}\bar{\varepsilon}\gamma\cdot\tilde{v}\tilde{\chi}\,,\\
\delta\omega_\mu{}^{ab} &= -2i\bar{\varepsilon}\gamma^{[a}\hat{\mathcal{R}}_\mu{}^{b]}(Q) - i\bar{\varepsilon}\gamma_\mu\hat{\mathcal{R}}^{ab}(Q)\\
&\quad- 2i\bar{\varepsilon}\gamma^{abcd}\psi_\mu\Big(\tilde{v}_{cd} + \tfrac{\mathcal{N}_I}{6\mathcal{N}}\hat{F}^I_{ab}(W)\Big) + 2i\bar{\varepsilon}\psi_\mu\tfrac{\mathcal{N}_I}{3\mathcal{N}}\hat{F}^I_{ab}(W)\\
&\quad- 4i\bar{\varepsilon}^i\gamma^{ab}\psi^j_\mu\tilde{t}_{ij} + \tfrac{4\mathcal{N}_{IJ}}{3\mathcal{N}}\Big((\bar{\varepsilon}\lambda^I)\bar{\psi}_\mu\gamma^{ab}\lambda^J - (\bar{\varepsilon}\gamma^{ab}\lambda^I)\bar{\psi}_\mu\lambda^J\Big)\,,
\end{aligned}
\tag{6·8}
$$

where $\mathcal{D}_\mu$ is the covariant derivative that is covariant only with respect to homogeneous transformations $\boldsymbol{M}_{ab}, \boldsymbol{U}^{ij}$ and $\boldsymbol{G}$, and the prime on $\boldsymbol{\Gamma}$ implies that $\boldsymbol{U}$-gauge field in $\boldsymbol{\Gamma}$ is removed: $\boldsymbol{\Gamma}\varepsilon^i = \boldsymbol{\Gamma}'\varepsilon^i - \gamma\cdot\tilde{V}^i_j\varepsilon^j$. Here we have also written the transformation law of the spin connection for convenience, although it is a dependent field.

The supersymmetry transformation laws of the vector multiplet are

$$
\begin{aligned}
\delta W^I_\mu &= -2i\bar{\varepsilon}\gamma_\mu\lambda^I + 2i\bar{\varepsilon}\psi_\mu M^I,\\
\delta M^I &= 2i\bar{\varepsilon}\lambda^I,\\
\delta\lambda^I_i &= \mathcal{P}^I{}_J(-\tfrac{1}{4}\gamma\cdot\hat{F}^J(W)\varepsilon_i - \tfrac{1}{2}\hat{\not{\mathcal{D}}}M^J\varepsilon_i + \tilde{Y}^J_{ij}\varepsilon^j) - \tfrac{M^I\mathcal{N}_{JK}}{3\mathcal{N}}2i\bar{\varepsilon}\lambda^J\lambda^K_i,\\
\delta\tilde{Y}^{Iij} &= 2i\bar{\varepsilon}^{(i}\mathcal{P}^I{}_J\hat{\not{\mathcal{D}}}\lambda^{Jj)} - i\bar{\varepsilon}^{(i}\gamma\cdot\tilde{v}\lambda^{Ij)} - i\tfrac{\mathcal{N}_J}{6\mathcal{N}}\bar{\varepsilon}^{(i}\gamma\cdot\hat{F}^J(W)\lambda^{Ij)}\\
&\quad+ 2i\bar{\varepsilon}^{(i}\tilde{t}^{j)}{}_k\lambda^{Ik} + 4i\bar{\varepsilon}\lambda^I\tilde{t}^{ij} - 2ig\bar{\varepsilon}^{(i}[M,\,\lambda]^{Ij)}\\
&\quad+ \tfrac{4\mathcal{N}_{JK}}{3\mathcal{N}}\bar{\varepsilon}\lambda^J\bar{\lambda}^{K(i}\lambda^{Ij)} - \tfrac{M^I\mathcal{N}_{JK}}{3\mathcal{N}}2i\bar{\varepsilon}\lambda^J\tilde{Y}^{Kij}.
\end{aligned}
\tag{6·9}
$$

Finally, the hypermultiplet transformation laws are given by

$$
\begin{aligned}
\delta\mathcal{A}^i_\alpha &= 2i\bar{\varepsilon}^i\xi_\alpha\,,\\
\delta\xi_\alpha &= -\hat{\not{\mathcal{D}}}\mathcal{A}^i_\alpha\varepsilon_i + \varepsilon_i gM_{\alpha\beta}\mathcal{A}^{i\beta} + \boldsymbol{\Gamma}'\varepsilon_i\mathcal{A}^i_\alpha + \Big(1 + \tfrac{\not{A}}{\alpha}\Big)\varepsilon_i\tilde{\mathcal{F}}^i_\alpha\,,\\
\delta\tilde{\mathcal{F}}^i_\alpha &= -2i\bar{\varepsilon}^i\Big(1 - \tfrac{\not{A}}{\alpha}\Big)^{-1}\Big(\hat{\not{\mathcal{D}}}\xi_\alpha + 2\tilde{\chi}_j\mathcal{A}^j_\alpha + gM_{\alpha\beta}\xi^\beta + 2g\lambda_{j\alpha\beta}\mathcal{A}^{j\beta}\\
&\qquad+ \tfrac{1}{2}\gamma\cdot\tilde{v}\,\xi_\alpha + \tfrac{2}{\alpha}\lambda^0_j\tilde{\mathcal{F}}^j_\alpha\Big) + \tfrac{2i}{\alpha}\bar{\varepsilon}\lambda^0\tilde{\mathcal{F}}^i_\alpha\,.
\end{aligned}
\tag{6·10}
$$

Here $gM = M^I gt_I$ and $g\lambda_i = \lambda^I_i gt_I$ include the $I = 0$ part with $gt_{I=0}$ as defined in Eq. (3·5), and the G covariantization for $I = 0$ in $\hat{\mathcal{D}}_\mu$ is understood to be $-A_\mu(gt_0)$, instead of the original central charge transformation $-\delta_Z(A_\mu)$. It is, however, interesting that the supersymmetry transformation rules for the latter two fields can be rewritten in slightly simpler forms if we refer to the original central charge transformation:

$$
\delta\xi_\alpha = -\hat{\not{\mathcal{D}}}_*\mathcal{A}^i_\alpha\varepsilon_i + \varepsilon_i gM_*\mathcal{A}^i_\alpha + \boldsymbol{\Gamma}'\varepsilon_i\mathcal{A}^i_\alpha\,,
$$

$$\delta\mathcal{F}^i_\alpha = -2i\bar{\varepsilon}^i\left(\hat{\not{\mathcal{D}}}_*\xi_\alpha + 2\tilde{\chi}_j\mathcal{A}^j_\alpha + gM'_{\alpha\beta}\xi^\beta + 2g\lambda'_{j\alpha\beta}\mathcal{A}^{j\beta} + \frac{1}{2}\tilde{v}\xi_\alpha\right) + \frac{4i}{\alpha}\bar{\varepsilon}^{(i}\lambda^{0j)}\mathcal{F}_{\alpha j}\,. \tag{6·11}$$

Here $\hat{\mathcal{D}}_*$ and M_* denote that the group action for the $I=0$ part is the original central charge transformation $\boldsymbol{Z}$; that is,

$$\hat{\mathcal{D}}_{*\mu} = \hat{\mathcal{D}}'_\mu - \delta_Z(A_\mu), \qquad gM_*\phi^\alpha = gM'^\alpha_\beta\phi^\beta + \delta_Z(\alpha)\phi^\alpha, \tag{6·12}$$

and the primes on $\hat{\mathcal{D}}'$, gM' and $g\lambda'_i$ denote that the $I=0$ parts are omitted.* The central charge transformation given in Eq. (I 4·5) can be rewritten in terms of our new variables, and reads, explicitly for $\mathcal{A}^i_\alpha$ and ξ_α, as $\delta_Z(\alpha)\mathcal{A}^i_\alpha = \mathcal{F}^i_\alpha$ and

$$\delta_Z(\alpha)\xi_\alpha = -\left(\hat{\not{\mathcal{D}}}_*\xi_\alpha + 2\tilde{\chi}_j\mathcal{A}^j_\alpha + gM'_{\alpha\beta}\xi^\beta + 2g\lambda'_{j\alpha\beta}\mathcal{A}^{j\beta} + \frac{1}{2}\tilde{v}\xi_\alpha\right) - \frac{2}{\alpha}\lambda^0_j\mathcal{F}^j_{\ \alpha}\,. \tag{6·13}$$

The last equation is equivalent to the central charge property of the $\boldsymbol{Z}$ transformation on $\mathcal{A}^i_\alpha$, $0 = \alpha\,[\delta_Z,\,\delta_Q(\varepsilon)]\mathcal{A}^i_\alpha = 2i\bar{\varepsilon}^i\delta_Z(\alpha)\xi_\alpha - \alpha\delta_Q(\varepsilon)(\mathcal{F}^i_\alpha/\alpha)$, which can also be rewritten in the following form, with $g\lambda_{j*} \equiv g\lambda'_j + (\lambda^0_j/\alpha)\delta_Z(\alpha)$:

$$\hat{\not{\mathcal{D}}}_*\xi_\alpha + 2\tilde{\chi}_j\mathcal{A}^j_\alpha + gM_*\xi_\alpha + 2g\lambda_{j*}\mathcal{A}^j_\alpha + \frac{1}{2}\tilde{v}\,\xi_\alpha = 0\,. \tag{6·14}$$

For convenience, we list here the explicit forms of the covariant derivatives appearing in these transformation laws:

$$\begin{aligned}
\mathcal{D}_\mu\varepsilon^i &= \left(\partial_\mu - \frac{1}{4}\gamma_{ab}\omega^{ab}_\mu\right)\varepsilon^i - \tilde{V}_\mu{}^i{}_j\varepsilon^j,\\
\hat{\mathcal{D}}_\mu\tilde{t}^{ij} &= \mathcal{D}_\mu\tilde{t}^{ij} - 4i\bar{\psi}^{(i}_\mu\tilde{\chi}^{j)} - i\bar{\psi}^{(i}_\mu\gamma\cdot\hat{\mathcal{R}}^{j)}(Q) - \frac{2i\mathcal{N}_{IJ}}{3\mathcal{N}}\left(\bar{\psi}^{(i}_\mu\hat{\not{\mathcal{D}}}M^I\lambda^{j)J} - \bar{\psi}_\mu\lambda^I\tilde{Y}^{Jij}\right),\\
\hat{\mathcal{D}}_\mu\tilde{v}_{ab} &= \mathcal{D}_\mu\tilde{v}_{ab} + 2i\bar{\psi}_\mu\gamma_{ab}\tilde{\chi} + \frac{i}{2}\bar{\psi}_\mu\gamma_{ab}\gamma\cdot\hat{\mathcal{R}}(Q) - \frac{i}{4}\bar{\psi}_\mu\gamma_{abcd}\hat{\mathcal{R}}^{cd}(Q) + i\bar{\psi}_\mu\hat{\mathcal{R}}_{ab}(Q)\,,\\
\mathcal{D}_\mu\tilde{t}^{ij} &= \partial_\mu\tilde{t}^{ij} - 2\tilde{V}^{(i}_\mu{}_k\tilde{t}^{j)k}, \qquad \mathcal{D}_\mu\tilde{v}_{ab} = \partial_\mu\tilde{v}_{ab} + 2\omega_{\mu[a}{}^c\tilde{v}_{b]c}\,,\\
\hat{\mathcal{D}}_\mu M^I &= \mathcal{D}_\mu M^I - 2i\bar{\psi}_\mu\lambda^I, \qquad \mathcal{D}_\mu M^I = \partial_\mu M^I - g[W_\mu,\,M]^I,\\
\hat{\mathcal{D}}_\mu\hat{F}_{ab}(W)^I &= \mathcal{D}_\mu\hat{F}_{ab}(W)^I - 4i\bar{\psi}_\mu\gamma_{[a}\hat{\mathcal{D}}_{b]}\lambda^I - 2i\bar{\psi}_\mu\hat{\mathcal{R}}_{ab}(Q)M^I - 4i\bar{\psi}_\mu\gamma\cdot\tilde{v}\gamma_{ab}\lambda^I\\
&\quad - 8i\bar{\psi}_\mu\lambda^I\tilde{v}_{ab} - 4i\bar{\psi}_\mu\gamma_{ab}\tilde{t}\lambda^I - \frac{\mathcal{N}_I}{3\mathcal{N}}i\bar{\psi}_\mu\gamma\cdot\hat{F}(W)^I\gamma_{ab}\lambda^I + \frac{8\mathcal{N}_{IJ}}{3\mathcal{N}}(\bar{\psi}_\mu\lambda^J)\bar{\lambda}^I\gamma_{ab}\lambda^K,\\
\mathcal{D}_\mu\hat{F}_{ab}(W)^I &= \partial_\mu\hat{F}_{ab}(W)^I - g[W_\mu,\,\hat{F}_{ab}(W)]^I + 2\omega_{\mu[a}{}^c\hat{F}_{b]c}(W)^I,\\
\hat{\mathcal{D}}_\mu\lambda^I_i &= \mathcal{D}_\mu\lambda^I_i + \mathcal{P}^I{}_J\left(\frac{1}{4}\gamma\cdot\hat{F}(W)^J\psi_{\mu i} + \frac{1}{2}\hat{\not{\mathcal{D}}}M^J\psi_{\mu i} - \tilde{Y}^J_{ij}\psi^j_\mu\right) + \frac{M^I\mathcal{N}_{JK}}{3\mathcal{N}}(2i\bar{\psi}_\mu\lambda^J)\lambda^K_i,\\
\mathcal{D}_\mu\lambda^I_i &= \left(\partial_\mu - \frac{1}{4}\gamma_{ab}\omega^{ab}_\mu\right)\lambda^I_i - \tilde{V}_{\mu ij}\lambda^{Ij} - g[W_\mu,\,\lambda_i]^I,\\
\hat{\mathcal{D}}_\mu\mathcal{A}^i_\alpha &= \mathcal{D}_\mu\mathcal{A}^i_\alpha - 2i\bar{\psi}^i_\mu\xi_\alpha\,, \qquad \mathcal{D}_\mu\mathcal{A}^i_\alpha = \partial_\mu\mathcal{A}^i_\alpha - \tilde{V}^i_{\mu j}\mathcal{A}^j_\alpha - gW_{\mu\alpha\beta}\mathcal{A}^{\beta i},\\
\hat{\mathcal{D}}_\mu\xi_\alpha &= \mathcal{D}_\mu\xi_\alpha + \hat{\not{\mathcal{D}}}\mathcal{A}^i_\alpha\psi_{\mu i} - \psi_{\mu i}gM\mathcal{A}^i_\alpha - \boldsymbol{\Gamma}'\psi_{\mu i}\mathcal{A}^i_\alpha - \left(1 + \frac{\not{A}}{\alpha}\right)\psi_{\mu i}\mathcal{F}^i_{\ \alpha}\,,\\
\mathcal{D}_\mu\xi_\alpha &= \left(\partial_\mu - \frac{1}{4}\gamma_{ab}\omega^{ab}_\mu\right)\xi_\alpha - gW_{\mu\alpha\beta}\xi^\beta.
\end{aligned} \tag{6·15}$$

---

* It may be worth mentioning that the transformation rules in Eq. (6·10) can also be rewritten equivalently by making the replacements $\tilde{\mathcal{F}}^i_\alpha,\ \hat{\mathcal{D}},\ gM,\ g\lambda_i \to \mathcal{F}^i_\alpha,\ \hat{\mathcal{D}}',\ gM',\ g\lambda'_i$.

The supercovariant curvatures $\hat{\mathcal{R}}_{\mu\nu}$ are obtained from $[\hat{\mathcal{D}}_a, \hat{\mathcal{D}}_b] = -\hat{\mathcal{R}}_{ab}{}^{\bar{A}}\boldsymbol{X}_{\bar{A}}$ as noted above, or can be read directly from the above transformation laws of the gauge field, (6·8), via the formulas (I 2·29), $\hat{\mathcal{R}}_{\mu\nu}{}^{\bar{A}} = 2\partial_{[\mu}h_{\nu]}^{\bar{A}} - h_\mu^{\bar{C}}h_\nu^{\bar{B}}f'_{\bar{B}\bar{C}}{}^{\bar{A}}$, and (I 2·24), $\delta h_\mu^{\bar{A}} = \partial_\mu\varepsilon^{\bar{A}} + \varepsilon^{\bar{C}}h_\mu^{\bar{B}}f_{\bar{B}\bar{C}}{}^{\bar{A}}$. Explicitly, they are given by

$$
\begin{aligned}
\hat{\mathcal{R}}_{\mu\nu}{}^i(Q) &= 2\mathcal{D}_{[\mu}\psi^i_{\nu]} + 2\gamma_{[\mu}\tilde{t}^i{}_j\psi^j_{\nu]} + \gamma_{[\mu ab}\psi^i_{\nu]}\tilde{v}^{ab} \\
&\quad + \frac{\mathcal{N}_I}{6\mathcal{N}}\gamma_{[\mu}\gamma\cdot\hat{F}^I(W)\psi^i_{\nu]} + \frac{4i\mathcal{N}_{IJ}}{3\mathcal{N}}\gamma_{[\mu}\lambda^{Ii}(\bar{\lambda}^J\psi_{\nu]})\,, \\
\hat{\mathcal{R}}_{\mu\nu}{}^{ij}(U) &= 2\partial_{[\mu}\tilde{V}^{ij}_{\nu]} - [\tilde{V}_\mu, \tilde{V}_\nu]^{ij} + 8i\bar{\psi}^{(i}_{[\mu}\gamma_{\nu]}\tilde{\chi}^{j)} + 2i\bar{\psi}^{(i}_{[\mu}\gamma_{\nu]ab}\hat{\mathcal{R}}^{abj)}(Q) \\
&\quad -4i\bar{\psi}^{(i}_\mu\gamma\cdot\left(\tilde{v} + \frac{\mathcal{N}_I}{4\mathcal{N}}\hat{F}^I(W)\right)\psi^{j)}_\nu + 6i\bar{\psi}_\mu\psi_\nu\tilde{t}^{ij} + \frac{8\mathcal{N}_{IJ}}{\mathcal{N}}(\bar{\psi}_{[\mu}\lambda^I)\bar{\psi}^{(i}_{\nu]}\lambda^{j)J}, \\
\hat{F}^I_{\mu\nu}(W) &= F^I_{\mu\nu}(W) + 4i\bar{\psi}_{[\mu}\gamma_{\nu]}\lambda^I - 2i\bar{\psi}_\mu\psi_\nu M^I.
\end{aligned}
\tag{6·16}
$$

§7. Compensators, gauged supergravity and scalar potential

7.1. *Independent variables*

We have labeled the vector multiplet $(M^I, W^I_\mu, \lambda^{Ii}, \tilde{Y}^{Iij})$ by the index I, taking $1+n$ values from 0 to n. However, it is only the vector component W^I_μ that actually has $1+n$ independent components. All the others have only n components, since the scalar components M^I satisfy the $\boldsymbol{D}$ gauge condition $\mathcal{N}(M) = 1$, and the fermion and auxiliary fields satisfy the constraints $\mathcal{N}_I\lambda^I = \mathcal{N}_I\tilde{Y}^I = 0$. Thus our parametrizations for them are redundant, although the gauge symmetry is realized linearly for these variables, and hence is more manifest there.

It is, of course, possible to parametrize these fields with independent variables, as was done by GST from the beginning in their on-shell formulation.[5] GST parametrized the manifold $\mathcal{M}$ of the scalar fields by ϕ^x with curved index $x = 1, \cdots, n$, and the fermions by λ^a with tangent index $a = 1, \cdots, n$. We can assign the same tangent index to our auxiliary fields and write $\tilde{Y}^a$.

The basic correspondence between the GST parametrization and ours is as follows:

$$
\begin{array}{ccc}
\text{GST parametrization} & & \text{our parametrization} \\
\mathcal{N} = C_{IJK}h^I(\phi)h^J(\phi)h^K(\phi) & \leftrightarrow & \mathcal{N} = c_{IJK}M^IM^JM^K \\
h^I(\phi) & = & -\sqrt{\frac{2}{3}}M^I|_{\mathcal{N}=1} \\
h_I(\phi) & = & -\frac{1}{\sqrt{6}}\mathcal{N}_I|_{\mathcal{N}=1}
\end{array}
\tag{7·1}
$$

From this, various geometrical quantities defined by GST can be translated into their counterparts in our formulation. The metric a_{IJ} of the ambient $1+n$ dimensional space is the

same as ours, and the metric g_{xy} of the scalar manifold $\mathcal{M}$, induced from a_{IJ}, is given by

$$g_{xy} \equiv a_{IJ} h^I_x h^J_y\,, \qquad \text{with} \quad h^I_x \equiv -\sqrt{\frac{3}{2}} h^I_{,x} = M^I_{,x}\,, \tag{7·2}$$

where $,x$ denotes differentiation with respect to ϕ^x. The indices $I, J, \cdots$ are raised and lowered by the metric a_{IJ} and its inverse a^{IJ}, and the indices $x, y, \cdots$ are raised and lowered by the metric g_{xy} and its inverse g^{xy}. The curved indices $x, y, \cdots$ are converted into the tangent indices $a, b, \cdots$ by means of the vielbein f^a_x and its inverse f^x_a, satisfying $f^a_x f^b_y \delta_{ab} = g_{xy}$ and $f^a_x f^b_y g^{xy} = \delta^{ab}$. Some useful relations are

$$\begin{aligned}
&h_{Ix} \equiv a_{IJ} h^J_x = \sqrt{\frac{3}{2}} h_{I,x}\,, \qquad h^I_a \equiv f^x_a h^I_x\,, \qquad T_{xyz} \equiv C_{IJK} h^I_x h^J_y h^K_z\,, \\
&h_I h^I = 1, \qquad h^x_I h^I_y = \delta^x_y, \qquad h_I h^I_x = h^I h^x_I = 0, \\
&a^{IJ} = g^{xy} h^I_x h^J_y + h^I h^J \quad \rightarrow \quad \delta^I_J = g^{xy} h^I_x h_{Jy} + h^I h_J = \mathcal{P}^I_J + \frac{M^I \mathcal{N}_J}{3\mathcal{N}}\,, \\
&a_{IJ} h^I_a h^J_{b,x} = \Omega_{xab} - \sqrt{\frac{2}{3}} T_{abx}\,, \qquad \tfrac{1}{2} a_{IJ,x} h^I_a h^J_b = \sqrt{\frac{2}{3}} T_{abc} f^c_x\,,
\end{aligned} \tag{7·3}$$

where Ω^{ab}_x is the 'spin-connection' of $\mathcal{M}$ defined as usual by $f^a_{[x,y]} + \Omega_{[y}{}^{ab} f^b_{x]} = 0$.

Now it is easy to rewrite our action and supersymmetry transformation laws in terms of the independent variables ϕ^x, λ^a_i and $\tilde{Y}^a_{ij}$. M^I is simply $-\sqrt{2/3} h^I(\phi)$, and the indices I and a of λ and $\tilde{Y}$ are mutually converted by

$$\lambda^a = h^a_I \lambda^I, \qquad \lambda^I = \mathcal{P}^I_J \lambda^J = h^I_a h^a_J \lambda^J = h^I_a \lambda^a. \tag{7·4}$$

For instance, the supersymmetry transformation laws (6·9) are rewritten as

$$\begin{aligned}
\delta W^I_\mu &= -2i h^I_a \bar{\varepsilon} \gamma_\mu \lambda^a - i\sqrt{6} h^I \bar{\varepsilon} \psi_\mu\,, \\
\delta \phi^x &= 2i f^x_a \bar{\varepsilon} \lambda^a, \\
\delta \lambda^a_i &= -\frac{1}{4} h^a_I \gamma\cdot\hat{F}^I(W) \varepsilon_i - \frac{1}{2} f^a_x \hat{\not{\mathcal{D}}} \phi^x \varepsilon_i + \tilde{Y}^a_{ij} \varepsilon^j - (\Omega^{ab}_x - \sqrt{\frac{2}{3}} T_x{}^{ab}) \delta\phi^x \lambda_{bi}\,, \\
\delta \tilde{Y}^a_{ij} &= 2i \bar{\varepsilon}_{(i} \hat{\not{\mathcal{D}}} \lambda^b_{j)} + 2i \bar{\varepsilon}_{(i} \left(\sqrt{\frac{3}{2}} h^I g L_I{}^a{}_b + (\Omega_x{}^a{}_b - \sqrt{\frac{2}{3}} T_x{}^a{}_b) \hat{\not{\mathcal{D}}} \phi^x \right) \lambda^b_{j)} \\
&\quad + i \frac{1}{\sqrt{6}} h_I \bar{\varepsilon}_{(i} \gamma\cdot\hat{F}^I(W) \lambda^a_{j)} - i \bar{\varepsilon}_{(i} \gamma\cdot\tilde{v} \lambda^a_{j)} - 2i \bar{\varepsilon}_{(i} \tilde{t}_{j)}{}^k \lambda^a_k + 4i (\bar{\varepsilon} \lambda^a) \tilde{t}_{ij} \\
&\quad - \frac{8}{3} \bar{\varepsilon} \lambda_b (\bar{\lambda}^a_{(i} \lambda^b_{j)}) - (\Omega_x{}^a{}_b - \sqrt{\frac{2}{3}} T_x{}^a{}_b) \delta\phi^x \tilde{Y}^b_{ij}\,.
\end{aligned} \tag{7·5}$$

Here, $L_I{}^a{}_b(\phi)$ is a function of ϕ^x appearing in the gauge transformation in the GST notation:

$$\begin{aligned}
&\delta_G(\theta) \phi^x = g K^x_I(\phi) \theta^I, \qquad \delta_G(\theta) \lambda^a = g L_I{}^a{}_b(\phi) \lambda^b \theta^I, \\
&K^x_I(\phi) = -\sqrt{\frac{3}{2}} h^x_K f_{JI}{}^K h^J, \\
&L_I{}^a{}_b(\phi) = -(\Omega_x{}^a{}_b - \sqrt{\frac{2}{3}} T_x{}^a{}_b) K^x_I + h^a_K f_{JI}{}^K h^J_b.
\end{aligned} \tag{7·6}$$

One can see that these transformation laws for the physical components W^I_μ, ϕ^x and λ^a_i agree with the GST result [5] if the auxiliary fields are replaced by their solutions [and $2\lambda^a$, $2\psi_\mu$, 2ε and $i\gamma_\mu(-i\gamma^\mu)$ here are identified with λ^a, ψ_μ, ε and $\gamma_\mu(\gamma^\mu)$ of GST.] One can also easily rewrite the action and see the agreement with GST for the on-shell part in the absence of the hypermultiplet.

In the case of the hypermultiplet, $\mathcal{A}^i_\alpha$ and ξ_α are independent variables off-shell. However, on-shell they become mutually dependent variables, since they satisfy the equations of motion $\mathcal{A}^2 = -2$ and $\mathcal{A}^{\bar{\alpha}}_i \xi_\alpha = 0$. Moreover, there remains the $SU(2)$ $\boldsymbol{U}$ gauge symmetry, with which three components of $\mathcal{A}^i_\alpha$ can be eliminated. (Thus at least four of the $\mathcal{A}^i_\alpha$ and two of the ξ_α can be eliminated. Generally, *compensator* components of the hypermultiplets can be eliminated by equations of motion and the gauge symmetries, as explained below.) It is possible to separate the variables even off-shell into genuine independent variables and other variables that vanish on-shell or can be eliminated by gauge fixing. Such independent variables are those used in the on-shell formulation, for instance, by Ceresole and Dall'Agata, [6] and they are formally very similar to the GST variables for vector multiplets. Hence, the rewriting of the hypermultiplet variables can be done in a manner similar to that in the vector multiplet case. The only complications in this case are the above mentioned separation of the on-shell (or gauge) vanishing variables, which depend on the number of the compensators (i.e., the structure of the hypermultiplet manifold).

7.2. *Compensator*

The $\boldsymbol{D}$ gauge fixing $\mathcal{N} = 1$ was necessary to obtain the canonical form of the Einstein-Hilbert term. Owing to the equation of motion $\mathcal{A}^2 + 2\mathcal{N} = 0$, this in turn implies that the relation

$$\mathcal{A}^2 \equiv \mathcal{A}^\alpha_i d_\alpha{}^\beta \mathcal{A}^i_\beta = -2 \tag{7·7}$$

must hold on-shell. But this is possible only if some components of the hypermultiplet $\mathcal{A}^\alpha_i$ have *negative metric.* [13] To see this, we recall the fact that the metric $d_\alpha{}^\beta$ of the hypermultiplet can be brought into the standard form [11]

$$d_\alpha{}^\beta = \begin{pmatrix} \mathbf{1}_{2p} & \\ & -\mathbf{1}_{2q} \end{pmatrix}. \qquad (p, q : \text{integer}) \tag{7·8}$$

We distinguish the first $2p$ components of the hypermultiplet $\mathcal{A}^\alpha_i$ with index $\alpha = 1, 2, \cdots, 2p$ from the rest of the $2q$ components, and use the indices a and $\underline{\alpha}$ to denote the former $2p$ and the latter $2q$ components, respectively. Also taking account of the hermiticity $\mathcal{A}^i_\alpha = -(\mathcal{A}^\alpha_i)^*$, the quadratic terms of the hypermultiplet read

$$\mathcal{A}^2 \equiv \mathcal{A}^\alpha_i d_\alpha{}^\beta \mathcal{A}^i_\beta = -(\mathcal{A}^a_i)^*(\mathcal{A}^a_i) + (\mathcal{A}^{\underline{\alpha}}_i)^*(\mathcal{A}^{\underline{\alpha}}_i) \equiv -|\mathcal{A}^a_i|^2 + |\mathcal{A}^{\underline{\alpha}}_i|^2,$$

$$\nabla^\mu \mathcal{A}_i^{\bar{\alpha}} \nabla_\mu \mathcal{A}_\alpha^i = -(\nabla^\mu \mathcal{A}_i^a)^* (\nabla_\mu \mathcal{A}_i^a) + (\nabla^\mu \mathcal{A}_i^{\underline{\alpha}})^* (\nabla_\mu \mathcal{A}_i^{\underline{\alpha}}). \tag{7·9}$$

Thus we see that the first $2p$ components $\mathcal{A}_i^a$ (corresponding to p quaternions) have negative metric and hence should not be physical fields. Indeed, they are so-called *compensator* fields, which are used to fix the extraneous gauge degrees of freedom. In the simplest case, $p = 1$, for instance, the compensator $\mathcal{A}_i^a$ has four real components, among which one component is already eliminated by the above condition (7·7). The remaining three degrees of freedom can also be eliminated by fixing the $SU(2)$ $\boldsymbol{U}$ gauge by the condition

$$\mathcal{A}_i^a \propto \delta_i^a \quad \rightarrow \quad \mathcal{A}_i^a = \delta_i^a \sqrt{1 + \tfrac{1}{2}|\mathcal{A}_i^{\underline{\alpha}}|^2} = -\mathcal{A}_a^i. \tag{7·10}$$

The target manifold $\mathcal{M}_Q$ of the scalar fields $\mathcal{A}_i^\alpha$ becomes $USp(2,2q)/USp(2) \times USp(2q)$ in this case. For $p \geq 2$, we need to have more gauge freedom to eliminate more negative metric fields. In particular, if we add vector multiplets which couple to the hypermultiplet but do not have their own kinetic terms, the corresponding auxiliary fields Y^{ij} do not have quadratic terms and act as multiplier fields to impose further constraints on the scalar fields $\mathcal{A}_i^\alpha$ on-shell.* For instance, it is known that the manifold $SU(2,q)/SU(2) \times SU(q) \times U(1)$ is realized for $p = 2$ by adding a $U(1)$ vector multiplet without a kinetic term. [14] (See Appendix B for a detailed explanation.) This manifold reduces to $SU(2,1)/SU(2) \times U(1)$ when $q = 1$, which is the manifold for the universal hypermultiplet appearing in the reduction of the heterotic M-theory on S^1/Z_2 to five dimensions. [4]

7.3. *$SU(2)_R$ or $U(1)_R$ gauging*

The so-called gauged supergravity is the supergravity in which the R symmetry G_R is gauged, and G_R may be either the $U(1)$ subgroup [2] or the entire $SU(2)$ group, [15] which act on the indices i of ψ_μ^i, λ^{Ii} and $\mathcal{A}_i^\alpha$. In our framework, this $SU(2)$ is already the gauge symmetry $\boldsymbol{U}$, whose gauge field is V_μ^{ij}. However, this gauge field V_μ^{ij} has no kinetic term and is an auxiliary field. To obtain a physical gauge field possessing a kinetic term, we must prepare another gauge field $W_\mu^{Ra}{}_b$ for G_R, under which only the compensator field $\mathcal{A}_i^a$ is charged:

$$\mathcal{D}_\mu \mathcal{A}_i^a = \partial_\mu \mathcal{A}_i^a - V_{\mu ij} \mathcal{A}^{aj} - g_R W_{R\mu}{}^a{}_b \mathcal{A}_i^b. \tag{7·11}$$

In this expression, we are assuming that the compensator has no group charges other than G_R and that $p = 1$ so that the index a runs over 1 and 2. The generator t_R of G_R is given by $i\vec{\sigma}^a{}_b$ in the case of $SU(2)_R$ with the Pauli matrix $\vec{\sigma}$, and by $i\vec{q} \cdot \vec{\sigma}^a{}_b$ with an arbitrary real

* The corresponding fermion component, the gaugino λ^i, also becomes a multiplier to impose a constraint on the hypermultiplet fermion fields ξ_α.

3-vector $\vec{q}$ of unit length $|\vec{q}\,| = 1$ in the case of $U(1)_R$:

$$W_{R\mu}{}^a{}_b = \begin{cases} \vec{W}_{R\mu} \cdot i\vec{\sigma}^a{}_b & \text{for } SU(2)_R, \\ W_{R\mu}\, i\vec{q} \cdot \vec{\sigma}^a{}_b & \text{for } U(1)_R. \end{cases} \tag{7·12}$$

It should be noted that the G_R gauging interferes with the possibility of a hypermultiplet mass term. Indeed, the symmetric tensor $\eta^{\alpha\beta}$ of the mass term (3·3) must be invariant under G, implying the constraint $[t_I, \eta] = 0$ on the matrix $\eta = (\eta^\alpha{}_\beta)$ for any generators t_I of G. In particular, for the generator t_R of G_R, which we are now assuming to rotate only the compensator components $\mathcal{A}_i^a$, this constraint implies that the 2×2 matrix $\eta^a{}_b$ in the compensator sector must commute with the above t_R. However, for the $G_R = SU(2)_R$ case, there is no such $\eta^a{}_b$ that commutes with all the Pauli matrices, so that the mass term cannot exist for the compensator. For the $G_R = U(1)_R$ case, on the other hand, the constraint allows $\eta^a{}_b \propto i\vec{q} \cdot \vec{\sigma}^a{}_b$. The mass term with this η yields, in the above $\mathcal{D}_\mu \mathcal{A}_i^a$, an additional 'central charge term' $-A_\mu (gt_{I=0})^a{}_b \mathcal{A}_i^b$, with gt_0 defined in Eq. (3·5). However, since $\eta^a{}_b \propto i\vec{q} \cdot \vec{\sigma}^a{}_b$, this term can be absorbed into the $-g_R W^a_{R\mu b} \mathcal{A}_i^b$ term, and Eq. (7·11) remains unchanged. Generally speaking, the $U(1)_R$-gauge field $W_{R\mu}$ is, of course, a member of our complete set of vectors $\{\, W_\mu^I \,\}$ and is given by a linear combination of the latter as

$$W_{R\mu} = V_I W_\mu^I, \tag{7·13}$$

with real coefficients V_I, which are non-vanishing only for the Abelian indices I. Therefore, if the mass term exists with $\eta^a{}_b = i\vec{q} \cdot \vec{\sigma}^a{}_b$, it is implied that the $I = 0$ coefficient V_0 is given by $g_R V_{I=0} = m/2$.

The gauge fields V_μ and $W_{R\mu}$ mix with each other. We redefine the $\boldsymbol{U}$ gauge field $V_{\mu}{}^i_j$ as

$$V_\mu^{\mathrm{N}ij} \equiv V_\mu^{ij} - g_R W_{R\mu}{}^{ij}, \tag{7·14}$$

while keeping the $SU(2)_R$ gauge field $W_{R\mu}$ intact. Then, noting the $SU(2)$ $\boldsymbol{U}$ gauge-fixing condition $\mathcal{A}_i^a \propto \delta_i^a$, we see that the compensator couples only to this new $SU(2)$ gauge field V_μ^{N} and no longer couples to the $SU(2)_R$ gauge field $W_{R\mu}$:

$$\mathcal{D}_\mu \mathcal{A}_i^a = (\delta_i^a \partial_\mu + V_{\mu}^{\mathrm{N}a}{}_i)\sqrt{1 + \tfrac{1}{2}|\mathcal{A}_i^{\underline{\alpha}}|^2}. \tag{7·15}$$

On the other hand, other fields carrying the original $SU(2)$ indices i now come to couple both to V_μ^{N} and $W_{R\mu}$, since V_μ should now be replaced by $V_\mu^{\mathrm{N}} + g_R W_{R\mu}$. Therefore the net effect of the $SU(2)_R$ [or $U(1)_R$] gauging is simply that 1) the auxiliary field V_μ is replaced by V_μ^{N}, and 2) the covariant derivative ∇_μ (or $\mathcal{D}_\mu$) should be understood to contain the $W_{R\mu}$ covariantization term $-\delta_R(W_{R\mu})$ if acting on the fields carrying the $SU(2)$ indices i. The previously derived action remains valid as it stands with this understanding.

7.4. *Scalar potential*

The scalar potential term can be read from the action (5·17) to be

$$V = \frac{1}{4}(a^{IJ} - M^I M^J)\mathcal{Y}_I^{ij}\mathcal{Y}_{Jij}|_{\text{bosonic part}} - \mathcal{A}_i^{\bar{\alpha}}(gM)^2{}_\alpha{}^\beta \mathcal{A}_\beta^i. \tag{7·16}$$

Here the first term has come from the elimination of the auxiliary fields Y^{Iij} of the vector multiplet and t^{ij} of the Weyl multiplet, and the second term from the hypermultiplet. Using Eq. (5·14) for $\mathcal{Y}_I^{ij}$, this potential can be rewritten in the form

$$\begin{aligned} V &= (a^{IJ} - M^I M^J)P_I^{ij}P_{Jij} + Q_i^{\bar{\alpha}}Q_\alpha^i \\ &= (a^{IJ} - M^I M^J)P_I^{ij}(P_J^{ij})^* - |Q_i^{a}|^2 + |Q_i^{\underline{\alpha}}|^2 \,, \end{aligned} \tag{7·17}$$

where

$$P_I^{ij} \equiv \mathcal{A}_\alpha^{(i} g t_I^{\bar{\alpha}\beta}\mathcal{A}_\beta^{j)} = d_\gamma{}^\alpha \mathcal{A}_\alpha^{(i} g t_I^{\gamma\beta}\mathcal{A}_\beta^{j)}, \qquad Q_i^\alpha \equiv g\delta_G(M)\mathcal{A}_i^\alpha = M^I(gt_I)^\alpha{}_\beta \mathcal{A}_i^\beta, \tag{7·18}$$

and we have used the hermiticity properties $(P_I^{ij})^* = P_{Iij}$ and $Q_\alpha^i = -(Q_i^\alpha)^*$. Since a_{IJ} is the metric of the vector multiplet, the first term $a^{IJ}P_I^{ij}(P_J^{ij})^*$ is positive definite. Negative contributions result from the terms $-|M^I P_I^{ij}|^2$ and $-|Q_i^a|^2$, the latter of which comes from the compensator component of the hypermultiplet.

Equation (7·17) is our general result for the scalar potential. Consider here the special case of $U(1)_R$-gauged supergravity in which $p = 1$ and $q = 0$; that is, there is a single (quaternion) compensator and no physical hypermultiplets. Then, the compensator $\mathcal{A}_i^a$ becomes simply a constant δ_i^a, by Eq. (7·10). If the compensator is charged only under the $U(1)_R$ in G, we have

$$\begin{aligned} P_I^{ij} &= \mathcal{A}_a^{(i} g t_I^{ab}\mathcal{A}_b^{j)} = g_R V_I \epsilon^{jk}(i\vec{q}\cdot\vec{\sigma})^i{}_k \,, \\ Q_i^a &= M^I V_I g_R (i\vec{q}\cdot\vec{\sigma})^a{}_i \,, \end{aligned} \tag{7·19}$$

and the scalar potential

$$\begin{aligned} V &= 2g_R^2(a^{IJ} - 2M^I M^J)V_I V_J = 2g_R^2(g^{xy}h_x^I h_y^J - 2h^I h^J)V_I V_J \\ &= g_R^2\left(\frac{9}{2}g^{xy}\frac{\partial W}{\partial\varphi^x}\frac{\partial W}{\partial\varphi^y} - 6W^2\right), \end{aligned} \tag{7·20}$$

where we have used the relations $a^{IJ} = g^{xy}h_x^I h_y^J + h^I h^J$ and $h^I = -\sqrt{2/3}M^I$ in Eqs. (7·1) and (7·3), and the definitions [16)]

$$W \equiv \sqrt{\frac{2}{3}}h^I V_I = -\frac{2}{3}M^I V_I \,, \qquad \frac{\partial W}{\partial\varphi^x} = -\frac{2}{3}h_x^I V_I = -\frac{2}{3}M_{,x}^I V_I \,. \tag{7·21}$$

This agrees with the result by GST.[5] If the physical vector multiplets are not contained in the system, the scalars φ^x do not appear either, and only the graviphoton with $I = 0$ exists. In this case $\mathcal{N} = c_{000}\alpha^3$, and $\alpha = M^{I=0}$ is determined to be $\sqrt{3/2}$ by the normalization requirement of the graviphoton kinetic term, $a_{00} = 1$. Then, $W = -\sqrt{2/3}V_0$, and hence the potential further reduces to

$$V = -4g_R^2 V_0^2 \,, \tag{7·22}$$

which agrees with the well-known anti-de Sitter cosmological term in the pure gauged supergravity.[2]

§8. Conclusion and discussion

In this paper, we have presented an action for a general system of Yang-Mills vector multiplets and hypermultiplet matter fields coupled to supergravity in five dimensions. The supersymmetry transformation rules were also found. We have given these completely in the off-shell formulation, in which all the auxiliary fields are retained. Our work can be considered an off-shell extension of the preceding work by GST[5] and its generalization by Ceresole and Dall'Agata.[6] [The latter authors also included 'tensor multiplet matter fields' (linear multiplets, in our terminology) with regard to which our system is less general.]

We have several applications in mind, such as compactifying on the orbifold S^1/Z_2 and/or adding D-branes to the system. Then, the power of the present off-shell formulation will become apparent. In particular, for the case of S^1/Z_2, it should be straightforward to determine how to couple the bulk fields to the fields on the boundary planes, since we can follow the general algorithm given by Mirabelli and Peskin for the case of the bulk Yang-Mills supermultiplet.[7] Indeed, this program has been started very recently by Zucker[17] using his off-shell formulation. He used a 'tensor multiplet' (linear multiplet) as a compensator for the five-dimensional (pure) supergravity and found that the 4D supergravity induced on the boundaries is a non-minimal version of $N = 1$ Poncaré supergravity with $16+16$ components containing one auxiliary spinor, which was presented by Sohnius and West long ago.[18] This non-minimal version is related to the new minimal version by the same authors.[19] Another version of $N = 1$ Poncaré supergravity, which is related to the usual minimal version,[20] will appear if we start with our 5D supergravity in which the compensator is a hypermultiplet.

Adding D-branes in the system is not so straightforward. First of all, a D-brane is a dynamical object whose position $X^\mu(x)$ in the bulk and its fermionic counterpart become a supermultiplet in 4D that realizes the bulk (local) supersymmetry non-linearly. The problem of identifying a supersymmetry transformation law for this multiplet and writing an invariant

action is already quite non-trivial, even in the case of rigid supersymmetry, and has long been studied by several authors. [21] Once this problem is settled, coupling the bulk supergravity to the fields on the D-brane should be easy also in this case. The off-shell formulation is essential in any case.

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Appendix A
— *A Representation Realizing Eq. (2·13)* —

The following is an example of the set of hermitian matrices $\{T_I\}$, realizing the property (2·13).

Let us prepare a representation vector ψ_i for each simple factor group G_i in G that gives a faithful representation R_i of G_i, and a suitable numbers of singlet vectors $\{\psi_\alpha\}$. Assigning to them suitable $U(1)_x$ charges also, we consider a representation of G whose representation vector is given by $\{\psi_j,\ \psi_\alpha\}$, which transforms as follows under $G=\prod_i G_i \times \prod_x U(1)_x$:

	under G_i	$U(1)_x$ charges
ψ_j	repr. R_j for $i=j$ and singlet for $i \neq j$	q_j^x
ψ_α	singlet	q_α^x

Let A_i be the generator label of the simple factor group G_i, a_i be the component label of the $\dim R_j$ vector $\psi_j=(\psi_j^{a_i})$, and $\rho_{R_i}(t_{A_i})=(\rho_{R_i}(t_{A_i})^{a_i}{}_{b_i})$ be the representation matrices of the generators acting on ψ_i in the representation R_i. Then the generators $t_I=(t_{A_i},t_x)$ of G are given in this representation by

$$\begin{aligned} t_{A_i}{}^{a_j}{}_{b_j} &= \delta_{ij}\rho_{R_i}(t_{A_i})^{a_i}{}_{b_i}, \qquad & t_{A_i}{}^{\alpha}{}_{\beta} &= 0, \\ t_x{}^{a_j}{}_{b_j} &= i\delta^{a_j}_{b_j} q_j^x, & t_x{}^{\alpha}{}_{\beta} &= i\delta^{\alpha}_{\beta} q_\alpha^x. \end{aligned} \tag{A·1}$$

The desired matrices T_I are given by $T_{A_i}=c_i t_{A_i}/i$ and $T_x=t_x/i$. Equations given by (2·13) to be satisfied are

$$\begin{aligned} G_i^3: &\quad 6c_{A_iB_iC_i} = -ic_i^3\,\mathrm{tr}\Big(\rho_{R_i}(t_{A_i})\{\rho_{R_i}(t_{B_i}),\rho_{R_i}(t_{C_i})\}\Big), \\ G_i^2U(1)_x: &\quad 3c_{A_iB_ix} = -c_i^2 q_i^x\,\mathrm{tr}\Big(\rho_{R_i}(t_{A_i})\rho_{R_i}(t_{B_i})\Big), \\ U(1)_xU(1)_yU(1)_z: &\quad 3c_{xyz} = \sum_i q_i^x q_i^y q_i^z \dim R_i + \sum_\alpha q_\alpha^x q_\alpha^y q_\alpha^z. \end{aligned}$$

The constants c_i and $U(1)_x$ charges q_i^x of ψ_i are fixed by the first and second equations, respectively. The third equation should be satisfied by adjusting the $U(1)_x$ charges q_α^x of ψ_α. Clearly, there are such solutions for q_α^x if there are sufficiently many ψ_α.

Appendix B
— $U(2,n)/U(2)\times U(n)$ *as a Hypermultiplet Manifold for* $p=2$ —

In this appendix we explain how the manifold $U(2,n)/U(2)\times U(n)$ appears as a target space manifold $\mathcal{M}_Q$ of the physical hypermultiplet scalar fields for the case $p=2$. This

is merely a detailed version of what was essentially shown long ago by Breitenlohner and Sohnius.[14)]

We consider the hypermultiplet $\mathcal{A}_i^\alpha$ in the standard representation, in which the matrices ${d_\alpha}^\beta$ and $\rho^{\alpha\beta}$ take the form[11)]

$$ {d_\alpha}^\beta = \begin{pmatrix} \mathbf{1}_{2p} & \\ & -\mathbf{1}_{2q} \end{pmatrix}, \qquad \rho^{\alpha\beta} = \rho_{\alpha\beta} = \begin{pmatrix} \epsilon & & \\ & \epsilon & \\ & & \ddots \end{pmatrix}. \quad (\epsilon \equiv i\sigma_2) \tag{B·1} $$

The hypermultiplet $\mathcal{A}_{\alpha i}$ is regarded as the $2(p+q) \times 2$ matrix

$$ \mathcal{A} = \begin{pmatrix} \\ \mathcal{A}_{\alpha i} \\ \\ \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \mathcal{A}_{2a-1,1} & & \mathcal{A}_{2a-1,2} \\ \mathcal{A}_{2a,1} & & \mathcal{A}_{2a,2} \\ & \vdots & \end{pmatrix}, \quad (a = 1, 2, \cdots, p+q) \tag{B·2} $$

which consists of $p+q$ 2×2-blocks. Each block can be identified with a quaternion, which is also mapped equivalently to a 2×2 matrix:

$$ \boldsymbol{q} \equiv q^0 + \boldsymbol{i}q^1 + \boldsymbol{j}q^2 + \boldsymbol{k}q^3 \quad \leftrightarrow \quad q^0\mathbf{1}_2 - i\vec{q}\cdot\vec{\sigma} = \begin{pmatrix} q^0 - iq^3 & -iq^1 - q^2 \\ -iq^1 + q^2 & q^0 + iq^3 \end{pmatrix}. \tag{B·3} $$

This is consistent with the hermiticity condition for the hypermultiplet:

$$ (\mathcal{A}_{\alpha i})^* = \mathcal{A}^{\alpha i} = \rho^{\alpha\beta}\epsilon^{ij}\mathcal{A}_{\beta j}, \quad \rightarrow \quad \mathcal{A}^\dagger = -\epsilon\mathcal{A}^{\mathrm{T}}\rho\,. \tag{B·4} $$

The group G transformation and $SU(2)$ $\boldsymbol{U}$ transformation act on $\mathcal{A}$ as

$$ \mathcal{A} \;\rightarrow\; \mathcal{A}' = g\mathcal{A}u^\dagger, \qquad g \in G, \quad u \in SU(2). \tag{B·5} $$

The G invariance of the quadratic form

$$ \mathcal{A}^{\alpha i}{d_\alpha}^\beta\mathcal{A}_{\beta j} \quad \leftrightarrow \quad \mathcal{A}^\dagger d\,\mathcal{A} = -\epsilon\mathcal{A}^{\mathrm{T}}\rho d\,\mathcal{A} \tag{B·6} $$

requires that the two conditions for $g \in G$,

$$ g^\dagger d\,g = d, \qquad g^{\mathrm{T}}\rho d\,g = \rho d\,, \tag{B·7} $$

be satisfied. The former implies $g \in U(2p, 2q)$ and the latter $g \in Sp(2p+2q; \boldsymbol{C})$, so that the group G must be a subgroup of $USp(2p, 2q) = U(2p, 2q) \cap Sp(2p+2q; \boldsymbol{C})$.

Now we consider the case $p = 2$, in which we gauge the $U(1)$ group, which acts on $\mathcal{A}$ as a phase rotation $e^{i\theta}$ for the odd rows and as $e^{-i\theta}$ for the even rows; that is, the generator is given by $T_3 = \sigma_3 \otimes \mathbf{1}_{p+q}$. We do not give a kinetic term for the vector multiplet $\boldsymbol{V}_3$ coupling

to this charge T_3. Then, the auxiliary field component Y_3^{ij} of this multiplet appears only in a linear form in the action: $2Y_{3j}^{\ i}\mathcal{A}^{\alpha i}d_\alpha{}^\beta T_{3\beta}{}^\gamma\mathcal{A}_{\gamma j} = 2\,\mathrm{tr}(Y_3\mathcal{A}^\dagger d\,T_3\mathcal{A})$. Thus it acts as a multiplier to impose the following three constraints on the hypermultiplet on-shell:

$$\mathrm{tr}(\sigma^a\mathcal{A}^\dagger d\,T_3\mathcal{A}) = 0 \qquad \text{for} \quad a = 1,2,3. \tag{B·8}$$

Moreover, we have one more constraint on-shell,

$$\mathrm{tr}(\mathcal{A}^\dagger d\,\mathcal{A}) = 2\,, \tag{B·9}$$

which comes from the equation of motion $\mathcal{A}^2 = -2\mathcal{N}$ and the $\boldsymbol{D}$ gauge fixing condition $\mathcal{N} = 1$. Recall that we have two quaternion compensators for the present $p = 2$ case. Hence there are eight (real) scalar fields with negative metric which should be eliminated. The above constraints eliminate four components, and we still have $SU(2)$ $\boldsymbol{U}$ symmetry acting on the index i and the $U(1)$ gauge symmetry for the charge T_3. We can eliminate the remaining four negative metric components by the gauge-fixing of these gauge symmetries, so that the theory is consistent.

The manifold of the hypermultiplet specified by these four constraints (B·8) and (B·9) have dimension $4(p+q) - 4 = 4 + 4q$, and it is seen to be $U(2,q)/U(2)\times U(q)$ as follows. First, we find that a representative element of $\mathcal{A}$ satisfying these constraints is given by

$$\mathcal{A}^{\mathrm{repr}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathbf{1}_2 \\ i\sigma_2 \\ \mathbf{0}_2 \\ \vdots \\ \mathbf{0}_2 \end{pmatrix} . \tag{B·10}$$

Second, to identify the manifold, it is sufficient to consider the half size $(p+q)\times 2$ complex matrix $\mathcal{A}_{\mathrm{odd}}$ that consists of the odd rows of $\mathcal{A}$ alone, since the even row elements are essentially the complex conjugates of the odd row elements, as stipulated by the reality condition of $\mathcal{A}$. In this half-size representation, we can see that unitary transformations of the above representative element,

$$\mathcal{A}_{\mathrm{odd}} = U\mathcal{A}_{\mathrm{odd}}^{\mathrm{repr}} = \frac{1}{\sqrt{2}}U\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ & \vdots & \\ 0 & 0 \end{pmatrix} , \qquad U \in U(p,q), \tag{B·11}$$

all satisfy the above constraints. But here, the subgroup $U(q) \subset U(p,q)$ rotating the lower q rows alone is inactive, so that the manifold of $\mathcal{A}_{\mathrm{odd}}$ given by this form is $U(p,q)/U(q)$ and

has dimension $(p+q)^2 - q^2 = p^2 + 2pq$. However, when $p = 2$, this dimension already equals the above dimension $4q+4$ for the hypermultiplet $\mathcal{A}$ specified by the constraints (B·8) and (B·9), and thus the manifold of the latter is proved to be $U(2,q)/U(q)$.

The manifold of the physical hypermultiplets is further reduced by the gauge fixing of $SU(2)$ and $U(1)$, and hence becomes $U(2,q)/U(q) \times U(2)$.

Note also that the gauge group G is reduced to a subgroup of $U(p,q)$ as a result of the gauging of $U(1)$. Indeed, the gauge transformation $g \in G$, compatible with the $U(1)$ symmetry, should commute with the $U(1)$ generator T_3: $gT_3 = T_3 g$. One can easily see that the group element g in $USp(2p, 2q)$ satisfying this condition further must have the form

$$g = \begin{pmatrix} U & 0 \\ 0 & U^{\mathrm{T}\,-1} \end{pmatrix}, \quad \text{on} \quad \begin{pmatrix} \mathcal{A}_{\rm odd} \\ \mathcal{A}_{\rm even} \end{pmatrix} \qquad U \in U(p,q). \tag{B·12}$$

This element clearly belongs to $U(p,q)$.

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